THE ZERO-DIVISOR GRAPH ASSOCIATED TO A SEMIGROUP

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Abstract. The zero-divisor graph of a commutative semigroup with zero is the graph whose vertices are the nonzero zero-divisors of the semigroup, with two distinct vertices adjacent if the product of the corresponding elements is zero. New criteria to identify zero-divisor graphs are derived using both graph-theoretic and algebraic methods. We find the lowest bound on the number of edges necessary to guarantee a graph is a zero-divisor graph. In addition, the removal or addition of vertices to a zero-divisor graph is investigated by using equivalence relations and quotient sets. We also prove necessary and sufficient conditions for determining when regular graphs and complete graphs with more than two triangles attached are zero-divisor graphs. Lastly, we classify several graph structures that satisfy all known necessary conditions but are not zero-divisor graphs.

1. Introduction

The zero-divisor graph was first introduced by Beck (1988) in the study of commutative rings, and later studied by Anderson and Livingston (1999). F. DeMeyer (DeMeyer et al., 2002) pioneered the use of zero-divisor graphs to study semigroups. A *graph* $G = (V, E)$ is a set of vertices V and edges E. Edges are unordered pairs of vertices, and we say vertices x and y are adjacent, denoted $x - y$, if there is an edge $(x, y) \in E$. The *degree* $d(x)$ of a vertex x is the number of vertices adjacent to x. Let S be a commutative semigroup with zero. The zero-divisor graph of S, denoted $\Gamma(S)$, is the graph with vertices corresponding to the nonzero zero-divisors of S, and distinct zero-divisors x and y are adjacent if and only if $xy = 0$. The zero-divisor graph provides a connection between graph theory and algebraic theory that aids in the study of the zero-divisor ideal. Many properties of $\Gamma(S)$ have been discovered. For example, $\Gamma(S)$ is connected and $diam(\Gamma(S)) \leq 3$ (i.e., there exists a path of length at most 3 between every two vertices) (Anderson and Livingston, 1999). For a graph G and $U \subseteq V(G)$, the *subgraph of G induced by U*, denoted $G[U]$, is the graph with $V(G[U]) = U$ and edge $(u, v) \in E(G[U])$ if and only if $(u, v) \in E(G)$. The *core* of a graph G is defined to be the largest induced subgraph of G such that every edge in the core is part of a cycle (see Figure 1). DeMeyer et al. (2002) proved that if $\Gamma(S)$ contains a cycle, then the core of G is a union of quadrilaterals and triangles and any vertex not in the core is an end (a vertex of degree 1). Define the *neighborhood* $N(x)$ of a vertex x to be the set of all vertices adjacent to x. DeMeyer and DeMeyer (2005) showed that for every pair of nonadjacent vertices x and y in $\Gamma(S)$, there must exist a vertex z such that $N(x) \cup N(y) \subseteq \overline{N(z)}$, where $\overline{N(z)} := N(z) \cup \{z\}$ is the *closure* of $N(z)$. The aforementioned necessary conditions are sufficient for graphs with fewer than 6 vertices to be zero-divisor graphs (DeMeyer and DeMeyer, 2005), but no general necessary and sufficient conditions for zero-divisor graphs are known.

A few classes of graphs are currently known to be zero-divisor graphs. The *complete graph* K_n has n vertices and all distinct vertices are adjacent. The *(disjoint) union* $G_1 \cup G_2$ of

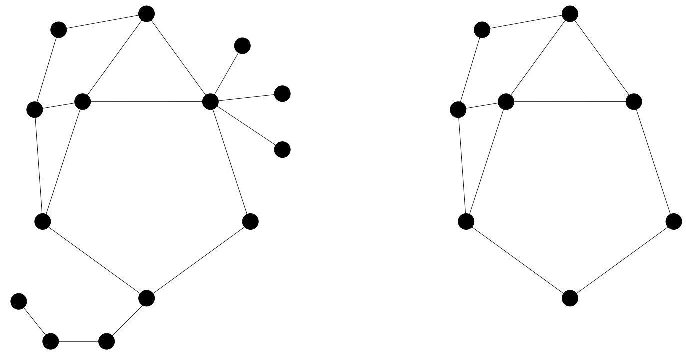

FIGURE 1. The graph G (left) and the core of G (right). Note that G contains a degree 2 vertex not in the core, so it is not a zero-divisor graph.

graphs G_1, G_2 with disjoint vertex sets is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$, and the *join* $G_1 \vee G_2$ of graphs G_1, G_2 is the union of the two graphs with additional edges (v_1, v_2) for all $v_1 \in G_1, v_2 \in G_2$. Let pK_n denote the union of p complete graphs (e.g., pK_1 represents p isolated vertices). Then the *complete n-partite graph* is $K_{p_1,p_2,\ldots,p_n} = \bigvee_{i=1}^{n} p_i K_1$. A 2-partite graph is called *bipartite.* $K_{1,p}$ is called a *star graph*, and the vertex adjacent to p other vertices is the *center* of the star graph. Note that $K_{1,1}$ is a star graph and has two centers. A graph is called the *refinement of a star graph* if there exists a vertex adjacent to all other vertices. Complete graphs, complete graphs together with an end, complete bipartite graphs, complete bipartite graphs together with an end, and refinements of star graphs are zero-divisor graphs (DeMeyer and DeMeyer, 2005). The semigroups S whose zero-divisor graph is an n-partite graph are described in DeMeyer et al. (To appear).

In this paper, we continue work on determining when an arbitrary graph is a zero-divisor graph and on characterizing all semigroups corresponding to a zero-divisor graph. We find the lowest possible bound on the number of edges $e(G)$ in a graph G that guarantees $G = \Gamma(S)$. Wu and Lu (2008) showed that for a zero-divisor semigroup S and vertex x of $\Gamma(S)$, removing all ends attached to x results in a sub-semigroup of S. We prove in Proposition 3.1 that if a vertex x is only adjacent to vertices of higher degree, then removing all vertices with neighborhoods that are a subset of $N(x)$ from $\Gamma(S)$ results in a zero-divisor graph, thus obtaining the previous result as a corollary. Wu and Lu (2006) showed that for $n \geq 4$, the complete graph K_n together with two end vertices has a unique corresponding zero-divisor semigroup, and K_n together with three end vertices has no corresponding semigroups. We continue this work by classifying when K_n with at least 3 degree 2 vertices attached is a zero-divisor graph. We additionally investigate when the removal or addition to $\Gamma(S)$ of vertices with identical neighborhoods results in another zero-divisor graph by using equivalence relations and quotient sets. We find that one can often determine if a graph is a zero-divisor graph by examining particular subgraphs. Let $I_k = \{x \in V(\Gamma(S)) \mid d(x) \geq k\} \cup \{0\}$ for a positive integer k. DeMeyer and DeMeyer (2005) proved that $\{I_k\}$ is a decreasing chain of ideals in S. We study I_Δ, the set of all vertices with maximum degree, and in doing so prove a necessary and sufficient condition for determining if a regular graph (a graph where all vertices have equal degree) is a zero-divisor graph. The complement $\overline{G}$ of graph G is the graph with the same vertices as G and with two vertices adjacent in $\overline{G}$ if and only if they are not adjacent in G. Jiang et al. (2006) showed that a graph is never a zero-divisor graph if its complement is a complete graph with at least 4 vertices that each has at least one end. In section 5, we generalize this result by showing that graphs with “modified star graphs” as their complements are never zero-divisor graphs.

Through this paper, all semigroups are multiplicative commutative semigroups with zero. We call a semigroup a zero-divisor semigroup if it consists solely of zero-divisors. The set of zero-divisors (including 0) of semigroup S, denoted $Z(S)$, forms an ideal. Note that $Z(S)$ is then a zero-divisor semigroup and $\Gamma(S) = \Gamma(Z(S))$. For $a \in S$, define the *annihilator set* of a as $ann(a) := \{x \in S \mid xa = 0\}$. Observe that $ann(a) = N(a) \cup \{0\}$ if $a^2 \neq 0$ and $ann(a) = \overline{N(a)} \cup \{0\}$ if $a^2 = 0$. Note that for a semigroup S and $a, b \in S$ with $ab \neq 0$, we have $N(a) \cup N(b) \subseteq \overline{N(ab)}$ since $ann(a) \cup ann(b) \subseteq ann(ab)$. Finally, we used a computer program to generate all non-isomorphic graphs with 6 vertices that are not zero-divisor graphs, but satisfy all previously known necessary conditions; these are shown in Figure 2.

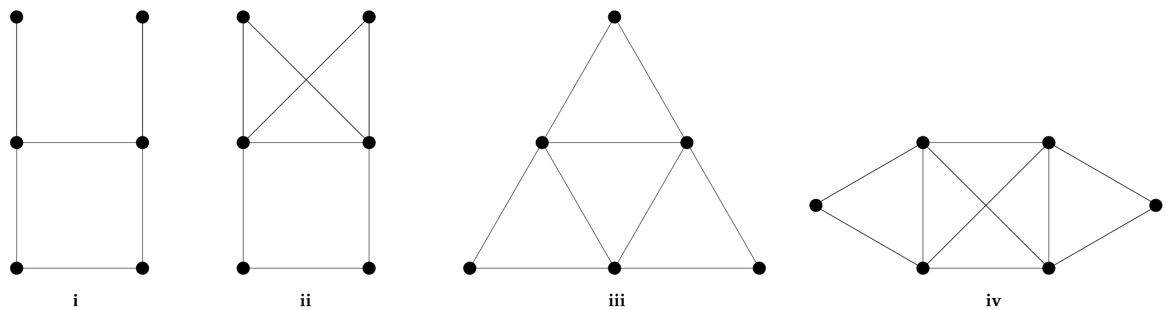

FIGURE 2. All graphs with 6 vertices that are not zero-divisor graphs but satisfy the mentioned necessary conditions.

2. EDGE COUNTING

In this section, we show that all connected graphs with a sufficient number of edges are zero-divisor graphs. It is straightforward to see that complete graphs K_n and complete graphs K_n with p edges removed are always zero-divisor graphs if p is very small relative to n. For this reason, we characterize graphs by the number of edges in their complements, and define $\mathcal{K}_{n,p}$ to be the set of all connected graphs with n vertices and $e(K_n) - p$ edges. Note that a complete graph K_n has $e(K_n) = \binom{n}{2} = \frac{n(n-1)}{2}$ edges.

For $p < n-1$, any graph in $\mathcal{K}_{n,p}$ has a disconnected complement. We therefore introduce a helpful lemma which allows us to break graphs with disconnected complements into smaller subgraphs.

Let the *disjoint graph* $\Gamma^*(S)$ of a semigroup S be the simple graph whose vertices are the nonzero elements (not necessarily zero-divisors) of S with distinct vertices x and y connected with an edge if $xy = 0$. The construction of $\Gamma^*(S)$ is very similar to the construction of $\Gamma(S)$. In fact, a graph G is the disjoint graph of a semigroup if and only if it is the union of the zero-divisor graph of a semigroup and some number of isolated vertices. To show this, observe that all zero-divisors in the disjoint graph must be contained in one connected component. The zero-divisors form an ideal, so we have the desired union. Conversely, let $G = \Gamma(S)$. Define $T = S \cup \{v_1, \ldots, v_n\}$ with the rules induced from S and $v_i x = v_1$ for all $x \in T$. Associativity holds, so T is a semigroup with $\Gamma^*(T) = G \cup nK_1$.

When not specified, the graph of a semigroup refers to the standard zero-divisor graph.

Lemma 2.1. *If graphs $\{G_\alpha\}$ are disjoint graphs of semigroups, then $G = \bigvee G_\alpha$ is a zero-divisor graph.*

Proof. Suppose each $G_\alpha = \Gamma(S_\alpha \cup \{0\})$, where $S_\alpha \cup \{0\}$ is a semigroup. Define $S = \bigcup S_\alpha \cup \{0\}$ with the rules induced from S_α and $ab = 0$ if a, b are not in the same S_α. Then $(ab)c = a(bc) = 0$ if a, b, and c are not in the same S_α; so S is associative and thus a semigroup with $\Gamma(S) = G$. ■

Lemma 2.2. *Let $G = \bigvee G_\alpha$ be a graph. Then each G_α is the disjoint graph of a semigroup if G is a zero-divisor graph, $|G_\alpha| > 1$, and G_α is not the refinement of a star graph for all α.*

Proof. Assuming $G = \Gamma(S)$, where S is a semigroup, let S_α be the set of elements of S associated with $V(G_\alpha)$. Take $a, b \in S_\alpha$ and suppose $ab = x \in S_\beta$ for $\alpha \neq \beta$. Since $|S_\beta| > 1$ and G_β is not the refinement of a star graph, there exists $y \in S_\beta$ such that $(ab)y = xy \neq 0$, but $a(by) = a(0) = 0$, contradicting associativity. Hence $ab \in S_\alpha$, and $S_\alpha \cup \{0\}$ is a semigroup with $\Gamma(S_\alpha \cup \{0\}) = G_\alpha$. ■

Corollary 2.3. *Let G be a connected graph which is not the refinement of a star graph, and let $H_1, \ldots, H_m$ be the connected components of the complement graph $\overline{G}$. Then G is a zero-divisor graph if and only if each $G[V(H_i)]$ is the disjoint graph of a semigroup.*

Corollary 2.4. *Any complete k-partite graph is a zero-divisor graph.*

We call an edge $u - v$ with $d(u) = d(v) = 1$ a *single component edge.*

The *degree sequence* of a graph is the non-increasing sequence of its vertex degrees.

Theorem 2.5.

(1) *All graphs in $\mathcal{K}_{n,p}$ for $p \leq \lceil n/2 \rceil + 1$ are zero-divisor graphs.*

(2) *For every integer p with $\lceil n/2 \rceil + 1 < p \leq e(K_n) - (n-1) = \frac{(n-1)(n-2)}{2}$, there exists a graph in $\mathcal{K}_{n,p}$ which is not a zero-divisor graph.*

Proof. (1) Assume $p \leq \lceil n/2 \rceil + 1$. Take $G \in \mathcal{K}_{n,p}$, which is a complete graph with p edges removed. If $p \leq \lfloor \frac{n-1}{2} \rfloor$, then G is missing at most $\frac{n-1}{2}$ edges. Thus removing p edges from K_n decreases the degree of at most $n-1$ distinct vertices, leaving at least one vertex with degree $n-1$. Hence G is the refinement of a star graph, which is the graph of a semigroup.

Now suppose $\lfloor \frac{n-1}{2} \rfloor < p \leq \lceil n/2 \rceil + 1$. For n even, observe that $p = \frac{n}{2}, \frac{n}{2} + 1$, and for n odd, $p = \frac{n+1}{2}, \frac{n+3}{2}$. If G has a vertex of degree $n-1$, then G is the graph of a semigroup. We next consider the four cases where G has no vertex of degree $n-1$.

Case 1. n is even, $p = \frac{n}{2}$. In order for G to not be the refinement of a star graph, G must have degree sequence $\{n-2, n-2, \ldots, n-2\}$, which implies that all p of the edges in $\overline{G}$ must be single component edges.

Case 2. n is even, $p = \frac{n}{2} + 1$. Then G can have degree sequence $\{n-3, n-3, n-2, \ldots, n-2\}$ or $\{n-4, n-2, \ldots, n-2\}$; so $\overline{G}$ has at least $p-4$ single component edges.

Case 3. n is odd, $p = \frac{n+1}{2}$. Then G can only have degree sequence $\{n-3, n-2, \ldots, n-2\}$; so $\overline{G}$ has exactly $p-2$ single component edges.

Case 4. n is odd, $p = \frac{n+3}{2}$. Then G can have degree sequence $\{n-5, n-2, \ldots, n-2\}$, $\{n-4, n-3, n-2, \ldots, n-2\}$, or $\{n-3, n-3, n-3, n-2, \ldots, n-2\}$; so $\overline{G}$ has at least $p-6$ single component edges.

Let k be the number of single component edges in $\overline{G}$. Removing all such components from $\overline{G}$, we have a graph W with $n-2k$ vertices and $p-k$ edges. Looking at the previous cases again,

Case 1. $n - 2k = n - 2p = 0$ and $p - k = 0$.

Case 2. $n - 2k \leq n - 2(p-4) = 6$ and $p - k = \frac{n}{2} + 1 - k$.

Case 3. $n - 2k \leq n - 2(p-2) = 3$ and $p - k = \frac{n+1}{2} - k$.

Case 4. $n - 2k \leq n - 2(p-6) = 9$ and $p - k = \frac{n+3}{2} - k$.

In all cases, $n - 2k \leq 9$ and $p - k \leq \lceil \frac{n-2k}{2} \rceil + 1$. We checked using a computer program that all (possibly disconnected) graphs with m vertices and $e(K_m) - q$ edges for for $m \leq 9$, $q \leq \lceil m/2 \rceil + 1$ are disjoint graphs of semigroups. Therefore $\overline{W}$ is the disjoint graph of a semigroup, and the complement of each single component edge is $2K_1$; so from Corollary 2.3, G is a zero-divisor graph.

(2) Assume $\lceil n/2 \rceil + 1 < p \leq e(K_n) - (n-1)$, which forces $n > 4$. Let G be a graph with n vertices $\{v_1, \ldots, v_n\}$ and $(n-1)$ edges $v_1 - v_3, v_1 - v_4, \ldots, v_1 - v_{n-1}, v_2 - v_4, v_2 - v_n$. Add edges to G so that G has $e(K_n) - \max(n, p)$ edges, but the n edges $v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n - v_1$ are not in G. If $p < n$, then also add in the edges $v_{4+2j} - v_{5+2j}$ for $0 \leq j < n - p$. Then v_1, v_2 are nonadjacent and from construction $\overline{N(v_1) \cup N(v_2)} = \{v_3, v_4, \ldots, v_n\}$. For $i = 1, 2, 3, n$, we have $v_{i-1}, v_{i+1} \notin \overline{N(v_i)}$, and for $3 < i < n$, either $v_{i-1} \notin \overline{N(v_i)}$ or $v_{i+1} \notin \overline{N(v_i)}$. Thus there is no v_i with $N(v_1) \cup N(v_2) \subseteq \overline{N(v_i)}$. Therefore G is not a zero-divisor graph.

Notice that $v_4 \in N(v_1) \cap N(v_2)$. Therefore the path $v_1 - v_4 - v_2$ connects v_1 and v_2. Since every vertex is contained in either $\overline{N(v_1)}$ or $\overline{N(v_2)}$, this shows that G is connected. Thus G is in $\mathcal{K}_{n,p}$. ■

Remark. The number of edges $e(K_n) = \binom{n}{2} = \langle n(n-1) \rangle 2$; so it follows from the theorem that a connected graph G is a zero-divisor graph if $e(G) \geq \binom{n}{2} - \lceil n/2 \rceil - 1$. The set $\mathcal{K}_{n,p}$ contains a refinement of a star graph for $p \leq e(K_n) - (n-1)$; so it cannot be determined from $e(G)$ whether G is a zero-divisor graph when $n - 1 \leq e(G) < \binom{n}{2} - \lceil n/2 \rceil - 1$. Lastly, any G with $e(G) < n - 1$ cannot be connected.

3. Removing and adding vertices

Corollary 2.3 shows that methods of identifying zero-divisor graphs from smaller subgraphs are very useful. We investigate when removing or adding vertices to zero-divisor graphs also result in zero-divisor graphs.

For the rest of the paper, all semigroups are assumed to be zero-divisor semigroups.

3.1. **Removing vertices.** The following proposition introduces a new semigroup ideal, which provides a sufficient condition for removing a set of vertices with similar neighborhoods from a zero-divisor graph.

Proposition 3.1. *Let $G = \Gamma(S)$, $a \in V$, and $T(a) = \{x : N(x) \subseteq N(a)\}$. If $N(x) - \overline{N(a)} \neq \emptyset$ for all $x \in N(a)$, then $(V - T(a)) \cup \{0\}$ is an ideal, and the induced subgraph $G[(V - T(a)) \cup \{a\}]$ is the disjoint graph of a semigroup.*

Proof. Take $x \in V - T(a)$ and $y \notin N(x)$. Then $N(x) \cup N(y) \subseteq \overline{N(xy)}$. If $|N(x) - N(a)| > 1$, then $N(xy) \not\subseteq N(a)$. Next, suppose that $|N(x) - N(a)| = 1$ and $N(xy) \subseteq N(a)$. Since $N(x) \subseteq \overline{N(xy)}$ but $N(x) \not\subseteq N(a)$, we must have $N(x) - N(a) = \{xy\}$. Therefore xy and x are adjacent; so $x \in N(xy) \subseteq N(a)$. Hence a and x are adjacent, and $a \in N(x) - N(a)$. By assumption, $N(x) - \overline{N(a)} \neq \emptyset$; so $a \neq xy$. This contradicts $|N(x) - N(a)| = 1$. Therefore $N(xy) \not\subseteq N(a)$ and $V - T(a)$ forms an ideal.

Let $A = \{x : N(x) = N(a)\}$. Suppose there does not exist an $x \in A$ such that $x^2 \in (V - T(a)) \cup \{0, x\}$. Then there must be a $C = \{x_1, \ldots, x_k\} \subseteq A$ with $k \geq 2$ such that $x_i^2 = x_{i+1}$ for $i = 1, \ldots, k-1$ and $x_k^2 = x_1$. Suppose $x = x_1 x_2 \cdots x_k \notin A$. Then $N(a) = N(x_1) \subseteq \overline{N(x)}$ implies either $N(a) \subsetneq N(x)$ since $x \notin A$ or $x \in N(a)$. We must have $x \in (V - T(a)) \cup \{0\}$ in both cases. Since $(V - T(a)) \cup \{0\}$ is an ideal, $x_1(x_1 x_2 \cdots x_k) = x_2 x_2 \cdots x_k = x_1 \in (V - T(a)) \cup \{0\}$, a contradiction. Therefore $x \in A$, but $(x_1 x_2 \cdots x_k)^2 = x_1^2 x_2^2 \cdots x_k^2 = x_2 \cdots x_k x_1 = x$, another contradiction. Thus there must exist an $x \in A$ with $x^2 \in (V - T(a)) \cup \{0, x\}$. Again, since $(V - T(a)) \cup \{0\}$ forms an ideal, $(V - T(a)) \cup \{0, x\}$ is a sub-semigroup. ∎

The conditions in the previous lemma are always satisfied if $d(x) > d(a)$ for all $x \in N(a)$ or if a is an end.

3.2. **Quotient semigroups and graphs.** We further investigate removing vertices with the same neighborhoods from zero-divisor graphs by using equivalence relations and quotients on semigroups and their associated graphs.

A *congruence* $\sim$ on a semigroup S is an equivalence relation with the property that $a \sim b,\ x \sim y \Rightarrow ax \sim by$ for all $a, b, x, y \in S$.

Let $\sim$ be an equivalence relation on S. Define the binary operation $\cdot$ on $S/{\sim}$ by $[a][b] := [ab]$ for $a, b \in S$, where ab is the product of a and b under the binary operation in semigroup S.

Theorem 3.2 (Howie, 1976, Theorem 5.3)**.** *$(S/{\sim}, \cdot)$ is a well-defined semigroup if and only if $\sim$ is a congruence on S.*

We define the *annihilator equivalence relation* $\sim_A$ on S by $a \sim_A b$ if $ann(a) = ann(b)$.

Theorem 3.3. *$\sim_A$ is a congruence on S.*

Proof. By inspection, $\sim_A$ is an equivalence relation. Suppose $a \sim_A b$ and $x \sim_A y$. Then $z \in ann(ax) \Rightarrow (ax)z = a(xz) = 0 \Rightarrow xz \in ann(a) = ann(b) \Rightarrow z \in ann(bx)$. Thus $ann(ax) \subseteq ann(bx)$, and similarly $ann(bx) \subseteq ann(by)$. Hence $ann(ax) \subseteq ann(by)$, and thus $ann(by) \subseteq ann(ax)$ by symmetry. Hence $ann(ax) = ann(by)$; so $ax \sim_A by$. Therefore $\sim_A$ is a congruence. ∎

Corollary 3.4. *$(S/{\sim_A}, \cdot)$ is a semigroup.*

We call the semigroup $S/{\sim_A}$ the *annihilator quotient semigroup* of S. Since x^2 may or may not equal 0 for $x \in S$, the zero-divisor graph of the annihilator quotient semigroup cannot be constructed from only $\Gamma(S)$. Annihilator sets are, however, very similar to neighborhood sets: $ann(x) = N(x) \cup \{0\}$ for $x \in S - \{0\}$ if $x^2 \neq 0$, and $ann(x) = \overline{N(x)} \cup \{0\}$ for

$x \in S - \{0\}$ if $x^2 = 0$. We show that neighborhood quotient graphs provide information on possible semigroups associated to the original graph.

Define the *open neighborhood equivalence relation* $\sim_O$ on the vertex set V by $u \sim_O v$ if $N(u) = N(v)$. Similarly define the *closed neighborhood equivalence relation* $\sim_C$ by $u \sim_C v$ if $\overline{N(u)} = \overline{N(v)}$. We construct the *open neighborhood graph* $\Xi_o(G)$ with $V(\Xi_o(G)) = V(G)/{\sim_O}$ and $[v]_O, [w]_O \in V(\Xi_o(G))$ are adjacent if and only if $v, w \in G$ are adjacent. Likewise, define the *closed neighborhood quotient graph* $\Xi_c(G)$ by using $\sim_C$.

An equivalent definition of $\Xi_o(G)$ is to construct $\Xi_o(G)$ by removing a vertex u from G if there is a vertex v with $N(u) = N(v)$, and repeating this process until no such vertex remains. The analogous definition is true for $\Xi_c(G)$. These alternate definitions show that $\Xi_o(G) \cong \Xi_o(\Xi_o(G))$ and $\Xi_c(G) \cong \Xi_c(\Xi_c(G))$. This behavior is mirrored in annihilator quotient semigroups.

Theorem 3.5. *Let S be a zero-divisor semigroup with $[0] = \{0\} \in S/{\sim_A}$, and let $[a], [b] \in S/{\sim_A}$. Then $ann([a]) = ann([b])$ if and only if $[a] = [b]$.*

Proof. Let $a, b \in S$ such that $ann([a]) = ann([b])$. Then $za = 0 \Rightarrow [z][a] = [0] \Rightarrow [z] \in ann([a]) = ann([b]) \Rightarrow [zb] = [z][b] = [0] = \{0\} \Rightarrow zb = 0$. Thus $ann(a) \subseteq ann(b)$ and by symmetry $ann(b) \subseteq ann(a)$; so $[a] = [b]$. The other direction is obvious. ∎

Corollary 3.6. *If S is a zero-divisor semigroup with $[0] = \{0\} \in S/{\sim_A}$, then $S/{\sim_A} \cong (S/{\sim_A})/{\sim_A}$.*

Observing that $S = V \cup \{0\}$ for a zero-divisor semigroup S, we extend $\sim_O$ and $\sim_C$ to equivalence relations on semigroups by setting $N(0) := V$ and $\overline{N(0)} := S$. Let $[x]_O$ denote an element of $S/{\sim_O}$ and $[x]_C$ denote an element of $S/{\sim_C}$. Note that $[0]_O = [0]_C = \{0\}$ under both equivalence relations because $a \notin N(a)$ and $\overline{N(a)} \subseteq V$ for $a \neq 0$.

If $\sim_O$ is a congruence, Theorem 3.2 implies $(S/{\sim_O}, \cdot)$ is a semigroup, and we observe that $\Gamma(S/{\sim_O}) = \Xi_o(\Gamma(S))$. Similarly if $\sim_C$ is a congruence, then $\Gamma(S/{\sim_C}) = \Xi_c(\Gamma(S))$.

Proposition 3.7. *Let $G = \Gamma(S)$. If $\Xi_o(G)$ is not a zero-divisor graph, then there exists an $x \in S - \{0\}$ with $x^2 = 0$. If G is not the refinement of a star graph, and $\Xi_c(G)$ is not a zero-divisor graph, then there exists an $x \in S - \{0\}$ with $x^2 \neq 0$.*

Proof. Suppose $x^2 \neq 0$ for all $x \in S - \{0\}$. Then $\sim_A$ and $\sim_O$ are equivalent; so $\Gamma(S/{\sim_A}) = \Xi_o(G)$, a contradiction. Suppose $x^2 = 0$ for all $x \in S$. There does not exist $y \in S$ with $ann(y) = ann(0)$ since G is not a refinement of a star graph. Therefore $\sim_A$ and $\sim_C$ are equivalent; so $\Gamma(S/{\sim_A}) = \Xi_c(G)$, a contradiction. ∎

Proposition 3.8. *If S is a semigroup and $\sim_O$ is a congruence, then $x^2 \neq 0$ for all $0 \neq x \in S$ with $|[x]_O| > 1$. If S is a semigroup and $\sim_C$ is a congruence, then $x^2 = 0$ for all $0 \neq x \in S$ with $|[x]_C| > 1$.*

Proof. Suppose $\sim_O$ is a congruence and there exists a nonzero $x \in S$ with $|[x]_O| > 1$ and $x^2 = 0$. Pick $y \in [x]_O$ not equal to x. Then $y \sim_O x \Rightarrow N(y) = N(x) \Rightarrow xy \neq 0$; so $[0]_O = [x^2]_O = [x]_O[x]_O = [x]_O[y]_O = [xy]_O$ contradicts $S/{\sim_O}$ being a semigroup.

Suppose $\sim_C$ is a congruence and there exists a nonzero $x \in S$ with $|[x]_C| > 1$ and $x^2 \neq 0$. Pick $y \in [x]_C$ not equal to x. Then $y \sim_C x \Rightarrow \overline{N(y)} = \overline{N(x)} \Rightarrow xy = 0$; so $[0]_C = [xy]_C = [x]_C[y]_C = [x]_C[x]_C$ contradicts $S/{\sim_C}$ being a semigroup. ∎

The following examples show when $\sim_O$ and $\sim_C$ may or may not be congruences. They also illustrate how the previous propositions may be used to determine whether $\sim_O$ and $\sim_C$ are congruences.

Example 3.9. Let G be the graph in Figure 3. Let S be the zero-divisor semigroup with $G = \Gamma(S)$ and rules $xy = x,\ x^2 = 0,\ y^2 = y,\ a^2 = a,\ b^2 = b$, and $c^2 = c$.
Then $[x]_O = \{x, y\}$ but $x^2 = 0$, so by Proposition 3.8, the relation $\sim_O$ is not a congruence on S. By direct calculation, $S/{\sim_C}$ is a semigroup, so $\sim_C$ is a congruence.

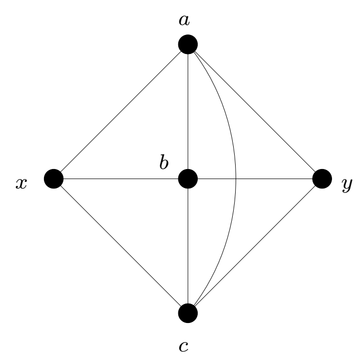

FIGURE 3. Graph G in Example 3.9.

Example 3.10. Let G be the graph in Figure 4. Let S be the zero-divisor semigroup with $G = \Gamma(S)$ and rules $a^2 = 0,\ b^2 = b,\ c^2 = c,\ x^2 = x,\ y^2 = y,\ ay = a,\ by = b$, and $cy = c$.
As $S \cong S/{\sim_O}$, we have that $S/{\sim_O}$ is a semigroup, so $\sim_O$ is a congruence on S. Since $[a]_C = \{a, b, c\}$ but $b^2 = b \neq 0$, Proposition 3.8 shows that $\sim_C$ is not a congruence on S.

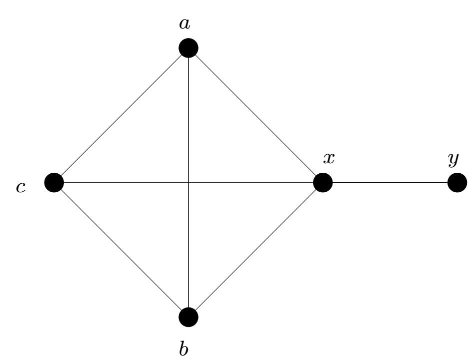

FIGURE 4. Graph G in Example 3.10.

Example 3.11. Let G be the graph in Figure 5. Let S be the zero-divisor semigroup with $G = \Gamma(S)$ and rules $x^2 = 0,\ y^2 = y$, and $a^2 = b^2 = c^2 = ab = bc = ac = a$.
By direction calculation, both $S/{\sim_O}$ and $S/{\sim_C}$ are semigroups, so $\sim_O$ and $\sim_C$ are both congruences on S.

Example 3.12. Let G be the graph in Figure 6. Let S be the zero-divisor semigroup with $G = \Gamma(S)$ and rules $x^2 = 0,\ y^2 = y,\ a^2 = 0,\ b^2 = b,\ c^2 = c,\ s^2 = s,\ t^2 = t,\ xt = x,\ yt = y,\ xy = x,\ at = a,\ bt = b$, and $ct = c$.
We see that $[x]_O = \{x, y\}$ but $x^2 = 0$, so by Proposition 3.8, the relation $\sim_O$ is not a congruence on S.
Also note that $[b]_C = \{a, b, c\}$ but $b^2 = b \neq 0$, so again by Proposition 3.8, the relation $\sim_C$ is not a congruence on S.

3.3. **Adding vertices.** As quotient graphs are formed by removing duplicate vertices, we investigate the "inverse" operation of adding duplicate vertices to a graph.

First we note that adding duplicate vertices to a zero-divisor graph does not always result in another zero-divisor graph.

Example 3.13. (1) The union of a 4-cycle $a_1 - a_2 - b_1 - b_2$ and one vertex c_1, where $N(c_1) = \{b_1, b_2\}$, is a zero-divisor graph. (2) The union of the same 4-cycle with $c_1, c_2, \dots, c_m$ vertices for $m \geq 2$ with $N(c_i) = \{b_1, b_2\}$ is never a zero-divisor graph.

Proof. (1) Let $S = \{a_1, a_2, b_1, b_2, c_1\}$ with rules

$\cdot$	0	a_1	a_2	b_1	b_2	c_1
0	0	0	0	0	0	0
a_1	0	a_1	0	b_1	0	b_1
a_2	0	0	a_2	0	b_2	b_2
b_1	0	b_1	0	0	0	0
b_2	0	0	b_2	0	0	0
c_1	0	b_1	b_2	0	0	0

One can check that S is associative and therefore a semigroup.
(2) Let G be the union of a 4-cycle $a_1 - a_2 - b_1 - b_2$ with vertices $c_1, \dots, c_m$ $(m \geq 2)$. Suppose $G = \Gamma(S)$. Then $\{a_2, b_1, b_2\} = N(a_1) \cup N(c_i) \subseteq \overline{N(a_1 c_i)} \Rightarrow a_1 c_i = b_1$, and similarly $a_2 c_i = b_2$. For $i \neq j$, $(a_1 c_i) c_j = b_1 c_j = 0 \Rightarrow c_i c_j \in ann(a_1)$, and similarly $c_i c_j \in ann(a_2)$. Therefore $c_i c_j \in ann(a_1) \cap ann(a_2) \subseteq \{a_1, a_2\}$. However $\{b_1, b_2\} \subseteq ann(c_i c_j)$, a contradiction. ∎

The next lemma presents a very useful condition for when duplicate vertices can be added.

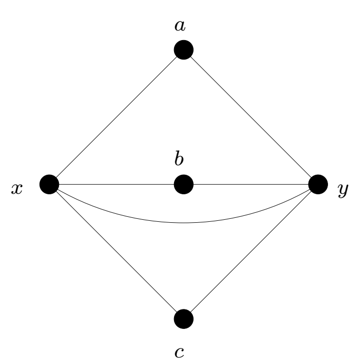

FIGURE 5. Graph G in Example 3.11.

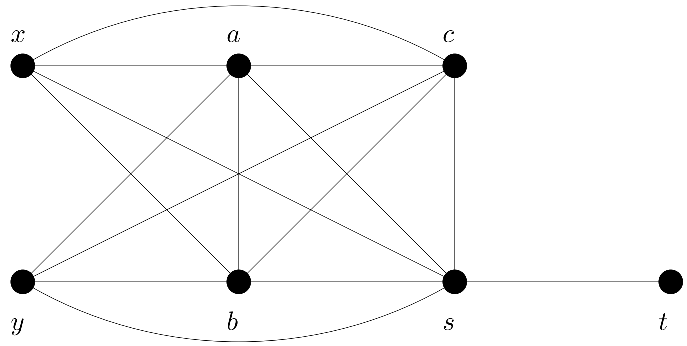

FIGURE 6. Graph G in Example 3.12.

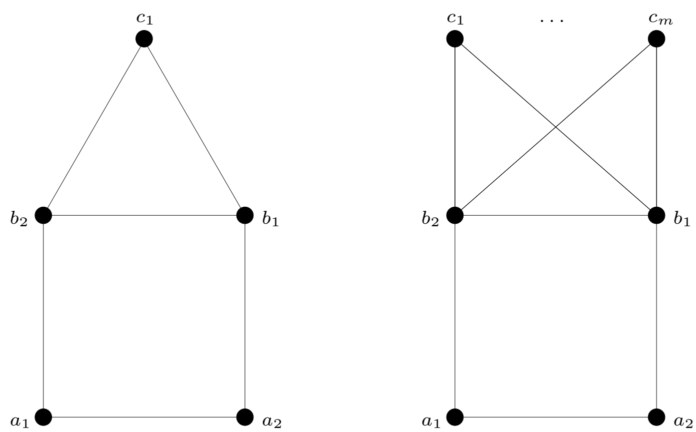

FIGURE 7. The graph on the left is a zero-divisor graph, but the graph on the right is not.

Lemma 3.14. *Let $G = \Gamma(S)$. For any $a \in S$, adding a vertex b to G with $N(b) = N(a)$ if $a^2 \neq 0$ or $N(b) = \overline{N(a)}$ if $a^2 = 0$ results in a zero-divisor graph.*

Proof. Let $S \cup \{b\}$ have rules induced from S. Define $bx = ax$ for all $x \in S - \{a\}$, and $b^2 = ab = a^2$. It is clear that $(xy)z = x(yz)$ if the rule $ab = a^2$ is not used. Otherwise if $xy = a$, we have $ab = (xy)b = x(yb) = x(ya) = (xy)a = a^2$. Note that xy never equals b. Lastly, $(ab)x = (a^2)x = a(ax) = a(bx) = b(ax)$. Therefore $S \cup \{b\}$ is a semigroup, and one can check that $\Gamma(S \cup \{b\})$ produces the desired graph. ■

Corollary 3.15. *Let $G = \Gamma(S)$. Then there exist infinitely many non-isomorphic zero-divisor graphs that contain G as a subgraph.*

Proof. With knowledge of the squares of elements in S, Lemma 3.14 allows us to "expand" the graph G by adding infinitely many additional vertices. ■

Corollary 3.16. *Let $G = \Gamma(S)$ with $a \in V$. If the graph formed by adding a vertex b to G with $N(b) = N(a)$ is not a zero-divisor graph, then $a^2 = 0$ in S.*

If the graph formed by adding a vertex b to G with $N(b) = \overline{N(a)}$ is not a zero-divisor graph, then $a^2 \neq 0$ in S.

Example 3.17. Let $G = \Gamma(S)$ be the graph with $V = \{a_1, a_2, b_1, b_2, c_1\}$ as in Figure 7. Adding a vertex c_2 to G with $N(c_2) = N(c_1)$ does not result in a zero-divisor graph, so $c_1^2 = 0$ in S from Corollary 3.16.

Corollary 3.18. *Let G be a connected graph that is not the graph of a semigroup. Then $\Xi_o(G)$ (with vertices $[x]_O, x \in V(G)$) is not the graph of a semigroup S that has $[x]_O^2 \neq 0$ for all $0 \neq [x]_O \in S$ with $|[x]_O| > 1$, and $\Xi_c(G)$ is not the graph of a semigroup S that has $[x]_C^2 = 0$ for all $0 \neq [x]_C \in S$ with $|[x]_C| > 1$.*

Corollary 3.19. *Let G be a connected graph. If $\Xi_o(G) = \Gamma(S)$ (with vertices $[x]_O, x \in V(G)$) for a semigroup S satisfying the property that $[x]_O^2 \neq 0$ for all nonzero $[x]_O \in S$ with $|[x]_O| > 1$, then G is a zero-divisor graph.*

If $\Xi_c(G) = \Gamma(S)$ for a semigroup S satisfying the property that $[x]_C^2 = 0$ for all nonzero $[x]_C \in S$ with $|[x]_C| > 1$, then G is a zero-divisor graph.

Example 3.20. Let G be the graph in Figure 8. Define semigroup $S = \{[0]_O, [x_1]_O, [y_1]_O, [z_1]_O\}$ with $[x_1]^2 = [x_1]$, $[y_1]^2 = [y_1]$, $[z_1]^2 = [z_1]$, and all other products equal to 0. As $\Xi_o(G) = \Gamma(S)$, Corollary 3.19 implies G is a zero-divisor graph.

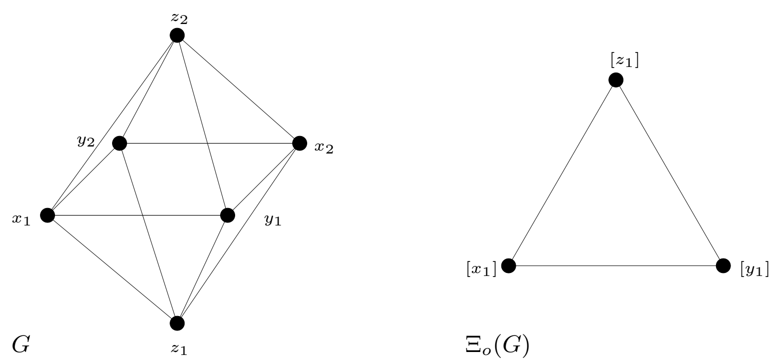

FIGURE 8. The octahedron G is a zero-divisor graph since $\Xi_o(G) = \Gamma(S)$, and S satisfies the property stated in Corollary 3.19.

Lemma 3.14 can only be used if one knows the value of a^2. The next lemma provides more specific graph conditions for when a^2 cannot equal 0.

Lemma 3.21. *Let $G = \Gamma(S)$ and $a \in V$. If there exists $y \notin N(a)$ such that for each $x \in \overline{N(a)}$, there exists $z \notin N(a)$ not equal to y that satisfies one of the following conditions:*

1. *$z \notin N(x)$ and $y - z$;*
2. *$z \notin N(x)$, $z \notin N(y)$, and for all w such that $N(y) \cup N(z) \subseteq \overline{N(w)}$, $w \in \overline{N(a)}$;*
3. *$z - x$, $z \notin N(y)$, and for all w such that $N(y) \cup N(z) \subseteq \overline{N(w)}$, $w \notin \overline{N(a)}$;*

then adding another vertex with the same neighbors as a results in the graph of a semigroup.

Proof. Suppose $a^2 = 0$. Then $a(ay) = a^2y = 0 \Rightarrow ay \in \overline{N(a)}$. Suppose $ay = x \in \overline{N(a)}$. Examining the 3 conditions, we have

1. $(ay)z = xz \neq 0$ and $a(yz) = 0$, or
2. $(ay)z \neq 0$ and $N(y) \cup N(z) \subseteq \overline{N(yz)} \Rightarrow a(yz) = 0$, or
3. $(ay)z = xz = 0$ and $yz \notin \overline{N(a)} \Rightarrow a(yz) \neq 0$.

Associativity does not hold, so we have a contradiction. Therefore $a^2 \neq 0$. Then from Lemma 3.14, we can add another vertex with same neighbors as a to G. ■

Theorem 3.22. *Let $G = \Gamma(S)$ be a graph with cycles.*

(1) *If a is an end adjacent to x, adding another end to x still results in a zero-divisor graph.*
(2) *Removing an end results in a zero-divisor graph.*

Proof. (1) If x is adjacent to every other vertex, then adding another end results in the refinement of a star graph. Otherwise, there exists $z \notin \overline{N(x)}$. Pick $y \in N(z)$. Then $y \neq x \Rightarrow y \notin N(a)$. Now G satisfies the conditions of Lemma 3.21, since $\overline{N(a)} = \{a, x\}$, $z \notin N(a)$, and $z \notin N(x)$.
(2) Let x have k ends. From Proposition 3.1, removing all ends adjacent to x except one results in a zero-divisor graph. Then using (1), we can add another $k-2$ ends to x, resulting in $k-1$ ends. ■

It has been determined in previous literature when adding ends to a complete graph results in a zero-divisor graph. We continue the study of adding vertices to complete graphs. The next theorem deals with complete graphs with triangles attached.

Theorem 3.23. *Let G be a graph with vertices $a_1, a_2, a_3, x_1, \ldots, x_n$ for $n \geq 4$. Suppose the induced subgraph $G[\{x_k\}_{k=1}^n] = K_n$, $d(a_i) = 2$, $N(a_i) \subseteq \{x_k\}_{k=1}^n$, and $N(a_i) \not\subseteq N(a_j)$ for all $i \neq j$. Then G is a zero-divisor graph if and only if $N(a_i) \cap N(a_j) \neq \emptyset$ for all i, j.*

We introduce two lemmas to prove the theorem.

Lemma 3.24. *Suppose $G = \Gamma(S)$ and I is an ideal of S. Then there cannot exist distinct $x_1, x_2 \in I$ and distinct $a_1, a_2, a_3 \notin I$ that satisfy the conditions:*

(1) $I \subseteq \overline{N(x_1)} \cap \overline{N(x_2)}$,
(2) $\{x_1, x_2\} \subseteq N(a_1)$,
(3) *a_2 and a_3 are not adjacent,*
(4) $\{x : N(a_2) \cup N(a_3) \subseteq \overline{N(x)}\} \subseteq I \cup (N(x_1) \cap N(x_2))$, *and*
(5) $(N(a_2) \cup N(a_3)) \cap (N(a_1) \cap I) = \emptyset$.

Proof. Assume for contradiction that the previous conditions hold. Then $x_1^2 a_1 = x_1(x_1 a_1) = 0 \Rightarrow x_1^2 \in ann(a_1) \cap I$. Suppose $x_1^2 \in ann(a_1) \cap I - \{0, x_1\}$. Then condition 5 requires that $x_1^2 a_2 \neq 0$ and $x_1^2 a_3 \neq 0$. Therefore $x_1 a_2 \in I - N(x_1)$, so $x_1 a_2 = x_1$ by condition 1. Similarly $x_1 a_3 = x_1$, and $(x_1 a_2) a_3 = x_1$. This is a contradiction since conditions 3 and 4 imply $a_2 a_3 \in I \cup N(x_1) \subseteq \overline{N(x_1)}$. Therefore $x_1^2 \in \{0, x_1\}$ and similarly $x_2^2 \in \{0, x_2\}$.

Case 1. Suppose $x_1^2 = x_1$ and $x_2^2 = x_2$.
Condition 5 implies a_2, a_3 are not adjacent to x_1, x_2 so $x_1^2 a_2 = x_1 a_2 \neq 0$. Then $x_1 a_2 \in I - N(x_1) \Rightarrow x_1 a_2 = x_1$. Similarly, $x_1 a_3 = x_1$. Conditions 3 and 4 imply $a_2 a_3 \in \overline{N(x_1)}$. Then $(x_1 a_2) a_3 = x_1 a_3 = x_1$ implies $a_2 a_3 = x_1$. By symmetry, however, we also get $a_2 a_3 = x_2$, a contradiction.

Case 2. Suppose $x_1^2 = 0$.
Conditions 3 and 4 imply $a_2 a_3 \in \overline{N(x_1)}$. Therefore $(x_1 a_2) a_3 = x_1(a_2 a_3) = 0 \Rightarrow x_1 a_2 \in N(a_3)$. Condition 5 then requires that $x_1 a_2 \notin N(a_1)$. However $(x_1 a_2) a_1 = a_2(x_1 a_1) = 0$, a contradiction.

We get a similar contradiction when $x_2^2 = 0$. Therefore we have contradictions in all cases; so G is not a zero-divisor graph. ∎

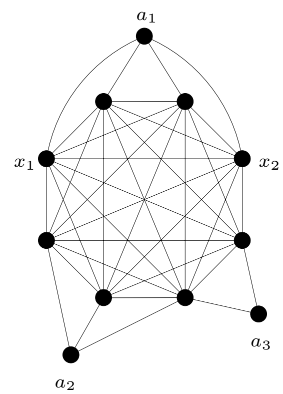

FIGURE 9. The graph for Example 3.25.

Example 3.25. The graph given in Figure 9 is not a zero-divisor graph. Let the ideal be I_8, and label a_1, a_2, a_3, x_1, x_2 as in the figure. The graph then satisfies the conditions of Lemma 3.24, so it cannot be a zero-divisor graph.

Lemma 3.26. *Let G be a graph with vertices $a_1, a_2, a_3, x_1, \ldots, x_n$ where $G[\{x_k\}_{k=1}^n] = K_n$,*

1. $N(a_i) \subseteq \{x_k\}_{k=1}^n$,
2. $N(a_i) \not\subseteq N(a_j)$ *for all* $i \neq j$,
3. $N(a_1) \cap N(a_2) \neq \emptyset$, $N(a_2) \cap N(a_3) \neq \emptyset$, *and* $N(a_1) \cap N(a_3) = \emptyset$, *and*
4. *one of:* $|N(a_1) - N(a_2)| > 1$, $|N(a_3) - N(a_2)| > 1$, *or* $N(a_2) \subseteq N(a_1) \cup N(a_3)$.

Then G is not a zero-divisor graph.

Proof. Pick $x_1 \in N(a_1) - N(a_2)$, which exists by condition 2. Then $a_2x_1 \in \overline{N(a_1)} \Rightarrow a_2x_1 \notin \overline{N(a_3)}$ by conditions 1 and 3. Hence $(a_2a_3)x_1 = a_3(a_2x_1) \neq 0$. Since $N(a_2) \cup N(a_3) \subseteq \overline{N(a_2a_3)}$, we cannot have $a_2a_3 = a_i$ by condition 2. Therefore $a_2a_3 = x_1$; so a_2a_3 cannot be defined if $|N(a_1) - N(a_2)| > 1$. Similarly a_1a_2 cannot be defined if $|N(a_3) - N(a_2)| > 1$. Now suppose $\{x_1 = a_2a_3\} = N(a_1) - N(a_2)$ and $\{a_1a_2\} = N(a_3) - N(a_2)$. Then $a_3x_1 \in \overline{N(a_1)}$, and by condition 3, $a_3x_1 \notin \overline{N(a_3)}$. Hence $a_3^2x_1 = a_3(a_3x_1) \neq 0$; so $a_3^2 \in \{x_1, a_2, a_3\}$. By assumption $a_1a_2 \notin N(a_2)$, and $a_1a_2 \neq a_2$ since $N(a_1) \cup N(a_2) \subseteq \overline{N(a_1a_2)}$. Therefore $a_2^2a_1 = a_2(a_1a_2) \neq 0 \Rightarrow a_2^2 \notin N(a_1)$. Again by condition 2, $a_2^2 \neq a_1$; so $a_2^2 \notin \overline{N(a_1)}$. Suppose $a_3^2 = a_2$. Then $a_2^2 = a_2a_3^2 = (a_2a_3)a_3 = x_1a_3 \in \overline{N(a_1)}$, a contradiction. Therefore $a_3^2 \in \{x_1, a_3\}$. Now pick $x_2 \in N(a_1) \cap N(a_2)$. In particular, $x_2 \neq x_1$. Then $a_3x_2 \in \overline{N(a_1)} \Rightarrow a_3x_2 \notin \overline{N(a_3)}$ by condition 3. Hence $a_3^2x_2 \neq 0 \Rightarrow a_3^2 \neq x_1$, and a_3^2 must equal a_3. We similarly conclude $a_1^2 = a_1$. We now have $(a_1a_3)a_2 = a_1(a_2a_3) = a_1x_1 = 0 \Rightarrow a_1a_3 \in \overline{N(a_2)}$. Using condition 2 again, $a_1a_3 \neq a_2$; so $a_1a_3 \in N(a_2)$. We also have $a_1(a_1a_3) = a_1a_3 \Rightarrow a_1a_3 \notin N(a_1)$, and $(a_1a_3)a_3 = a_1a_3 \Rightarrow a_1a_3 \notin N(a_3)$. Therefore $N(a_2) \not\subseteq N(a_1) \cup N(a_3)$. ∎

Example 3.27. The graph G given in Figure 10 is not a zero-divisor graph. Labeling a_1, a_2, a_3 as in the figure, we observe that $V(G) = V(K_9) \cup \{a_1, a_2, a_3\}$ and G satisfies the conditions in Lemma 3.26, so it is not a zero-divisor graph.

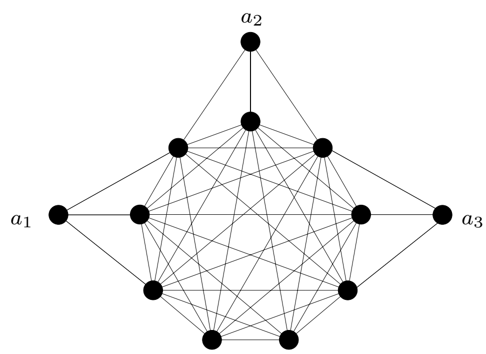

FIGURE 10. Graph for Example 3.27.

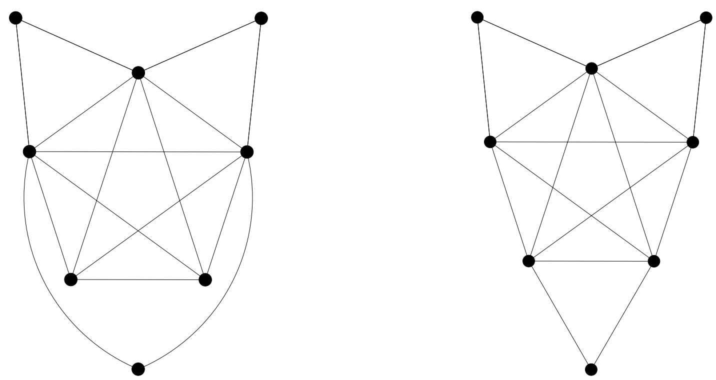

FIGURE 11. Theorem 3.23 shows that the graph on the left is a zero-divisor graph, while the graph on the right is not.

Proof of Theorem 3.23. ($\Rightarrow$) Suppose without loss of generality that $N(a_1) \cap N(a_3) = \emptyset$. If $N(a_1) \cap N(a_2) = \emptyset$ or $N(a_2) \cap N(a_3) = \emptyset$, then the graph is not a zero-divisor graph from Lemma 3.24 using the ideal I_n. Therefore $N(a_1) \cap N(a_2) \neq \emptyset$ and $N(a_2) \cap N(a_3) \neq \emptyset$. Now G satisfies Lemma 3.26, so it is again not a zero-divisor graph. Therefore $N(a_i) \cap N(a_j) \neq \emptyset$ for all i, j.

($\Leftarrow$) Assume $N(a_i) \cap N(a_j) \neq \emptyset$ for all i, j, and G is not the refinement of a star graph. Then let $N(a_1) \cap N(a_2) = \{x_3\}$, $N(a_1) \cap N(a_3) = \{x_2\}$, and $N(a_2) \cap N(a_3) = \{x_1\}$. Define S with operations $a_i^2 = a_i,\ a_1a_2 = a_1a_3 = a_2a_3 = x_n,\ x_i^2 = 0$ for $i \neq n,\ x_n^2 = x_n,\ a_ix_j = x_i$ for $x_j \notin N(a_i) \cup \{x_n\}$, and $a_ix_n = x_n$. One can check that S is a semigroup with $G = \Gamma(S)$. ■

Corollary 3.28. *Adding at least 4 triangles to a complete graph with at least 4 vertices results in a zero-divisor graph if and only if the graph is a refinement of a star graph.*

Proof. Let G be a complete graph with $m \geq 4$ triangles which is not the refinement of a star graph. We can always remove $m - 3$ triangles from G to obtain a graph that does not satisfy the conditions in Theorem 3.23 and is therefore not a zero-divisor graph. It follows from Proposition 3.1 that G is also not the graph of a semigroup. ■

Example 3.29. See Figure 11.

4. Vertex degrees

Many of the results in the previous section involved vertex neighborhoods, so we investigate zero-divisor graphs based on the order of these neighborhoods, or in other words the degrees of the vertices.

Let Δ, δ denote the maximum and minimum degrees of a graph, respectively.

Lemma 4.1. *Let $G = \Gamma(S)$ and $0 \neq x \in S$ with $d(x) = \Delta$. Then $N(y) \subseteq N(x)$ for all $y \notin N(x)$.*

Proof. Since x and y are not adjacent, $N(x) \cup N(y) \subseteq \overline{N(xy)} - \{x\}$. If $x \in N(xy)$, then $|N(x) \cup N(y)| \leq |\overline{N(xy)} - \{x\}| \leq \Delta$. Then $N(y) \subseteq N(x)$ since $|N(x)| = \Delta$. Otherwise if $xy \notin N(x)$, then $N(x) \subseteq N(xy)$. Since $|N(xy)| \leq \Delta = |N(x)|$, we have $N(x) = N(xy)$. Because x and y are not adjacent, $y \notin N(x) = N(xy) \Rightarrow xy \notin N(y)$. Therefore $N(y) \subseteq N(xy) = N(x)$. ■

Proposition 4.2. *If $\lceil \delta \cdot (|G| - \Delta - 1)/\Delta \rceil + 1 > \Delta$, then G is not a zero-divisor graph.*

Proof. Pick $x \in V$ with $d(x) = \Delta$. Then $V = \{x, y_1, \ldots, y_{n-\Delta-1}, w_1, \ldots, w_\Delta\}$ where $n = |G|$, $x - w_i$ for $i = 1, \ldots, \Delta$, and x and y_i are not adjacent for $i = 1, \ldots, n - \Delta - 1$. From Lemma 4.1, we have the inclusion $N(y_i) \subseteq N(x)$; so y_i can only be adjacent to w_j's. Each y_i must be adjacent to at least δ vertices (for $\delta \cdot (n - \Delta - 1)$ total adjacencies), so by the pigeonhole principle, there must exist w_i with degree $d(w_i) \geq \lceil \delta \cdot (n - \Delta - 1)/\Delta \rceil + 1 > \Delta$, a contradiction. ■

Recall that $G[I_\Delta]$ is the induced subgraph on G of all vertices of maximum degree.

Proposition 4.3. *If $G = \Gamma(S)$, then $G[I_\Delta]$ is either connected or the graph of isolated vertices. If $G[I_\Delta]$ are isolated vertices, at most one $x \in I_\Delta - \{0\}$ has $x^2 = 0$.*

Proof. Suppose $G[I_\Delta]$ has more than one connected component, vertices a and b are in the same component with $a - b$, and a, b are both not adjacent to c. Then from Lemma 4.1, $N(c) = N(a)$ and $N(c) = N(b) \Rightarrow a$ and b are not adjacent, a contradiction.

Suppose $G[I_\Delta]$ are isolated vertices and there exist $x, y \in I_\Delta - \{0\}$ with $x^2 = y^2 = 0$. Then $x^2 y = 0 \Rightarrow xy \in ann(x)$. Since I_Δ is an ideal, $xy \in ann(x) \cap I_\Delta = \{0, x\}$. As xy is nonzero, $xy = x \Rightarrow (xy)y = xy = x$ contradicts $xy^2 = 0$. ■

A k-regular graph is a graph where $\delta(G) = \Delta(G) = k$. Many well studied graphs are regular graphs, and the next theorem provides a necessary and sufficient condition for determining which regular graphs are zero-divisor graphs.

Theorem 4.4. *A connected k-regular graph G is a zero-divisor graph if and only if $n - k \mid n$ and $G = \bigvee^{n/(n-k)} (n-k)K_1$.*

Proof. ($\Rightarrow$) Define the relation $\sim$ on V by $u \sim v$ if u and v are non-adjacent in G. The relation $\sim$ is clearly reflexive and symmetric since G is a simple graph. Since G is regular, every vertex has maximum degree. Then Lemma 4.1 says that $u \sim v$ and $v \sim w \Rightarrow N(u) \subseteq N(v) \subseteq N(w) \Rightarrow u \sim w$; so $\sim$ is transitive. Therefore $\sim$ is an equivalence relation.

Set $Q = V/{\sim}$ and consider $[v] \in Q$. Since G is regular, v must be adjacent to k vertices and therefore not adjacent to $n - k$ vertices, including itself. Hence $[v]$ has order $n - k$. Since Q forms a partition of V, $n - k \mid n$. It is easy to see that each $[v]$ corresponds to one

$(n-k)K_1$, so $G = \bigvee^{n/(n-k)}(n-k)K_1$.
($\Leftarrow$) Follows from Corollary 2.4. ■

Example 4.5. The Petersen graph (see Godsil and Royle, 2001) is a 3-regular graph with 10 vertices. Since 7 does not divide 10, the Petersen graph is not a zero-divisor graph.

5. Special graphs

5.1. **Removing modified star graphs.** We show that a particular collection of graphs characterized by their complements are never zero-divisor graphs.

Theorem 5.1. *Let G be a graph. Suppose the complement $\overline{G}$ is the graph of m star graphs with centers $c_1, c_2, \ldots, c_m$ where c_i is adjacent to $\beta_i \geq 2$ degree 1 vertices $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,\beta_i}$ and at least two c_i vertices are adjacent (see Figure 12). Then G is not a zero-divisor graph.*

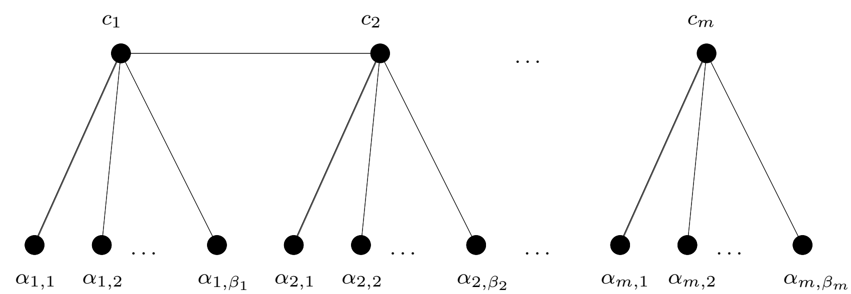

Figure 12. $\overline{G}$ satisfying conditions of Theorem 5.1.

Proof. Suppose $G = \Gamma(S)$. Without loss of generality, let c_1, c_2 be non-adjacent $\Rightarrow c_1c_2 \neq 0$. We then consider the two cases $c_1c_2 = c_k$ and $c_1c_2 = \alpha_{k,h}$.

Case 1. Suppose $c_1c_2 = c_k$. Then $\{\alpha_{i\neq 1,j}\} \subseteq N(c_1)$ and $\{\alpha_{i\neq 2,j}\} \subseteq N(c_2)$; so $\{\alpha_{i,j}\} \subseteq N(c_1) \cup N(c_2) \subseteq \overline{N(c_1c_2)} = \overline{N(c_k)}$, a contradiction.

Case 2. Suppose $c_1c_2 = \alpha_{k,h}$. Without loss of generality, let $k \neq 2$. Fix l such that $1 \leq l \leq \beta_2$. Then $(c_1c_2)\alpha_{2,l} = \alpha_{k\neq 2,h}\alpha_{2,l} = 0 \Rightarrow 0 \neq c_2\alpha_{2,l} \in ann(c_1) \Rightarrow c_2\alpha_{2,l} \neq \alpha_{1,j}$. We notice that $N(\alpha_{i,j}) \subseteq \overline{N(x\alpha_{i,j})}$. Hence $x\alpha_{i,j} \neq c_m$ for $x \in S$; so the set $I = \{\alpha_{i,j}\} \cup \{0\}$ forms an ideal in S. Therefore $c_2\alpha_{2,l} \in I \cap ann(c_1) = \{\alpha_{i\neq 1,j}\}$. For $i > 2$, we have $c_i(c_2\alpha_{2,l}) = (c_i\alpha_{2,l})c_2 = 0 \Rightarrow c_2\alpha_{2,l} \neq 0 \in ann(c_i) \Rightarrow c_2\alpha_{2,l} \neq \alpha_{i>2,j}$. Hence $c_2\alpha_{2,l} = \alpha_{2,j} \Rightarrow c_2(c_2\alpha_{2,l}) = c_2\alpha_{2,j} \neq 0 \Rightarrow c_2^2 \notin ann(\alpha_{2,l}) \Rightarrow c_2^2 \notin (I - \{\alpha_{2,l}\}) \cup \{c_{i\neq 2}\} \Rightarrow c_2^2 \in \{c_2, \alpha_{2,l}\}$. Additionally, $(c_1c_2)c_2 = \alpha_{k,h}c_2 = 0 \Rightarrow c_2^2 \in ann(c_1)$. Since c_1 and c_2 are non-adjacent in G, we have $c_2 \notin ann(c_1)$. Therefore $c_2^2 = \alpha_{2,l}$. This is true for all $1 \leq l \leq \beta_2$ and since $\beta_2 \geq 2$, we have a contradiction.

There is a contradiction in both cases, so G cannot be a zero-divisor graph. ■

Example 5.2. Let G be the graph in Figure 13. Note that G is one of the graphs mentioned in Figure 2. The complement $\overline{G}$ is the union of two star graphs with adjacent centers, so Theorem 5.1 implies G is not the graph of a semigroup.

The following theorem is a corollary to a result by Jiang et al. (2006).

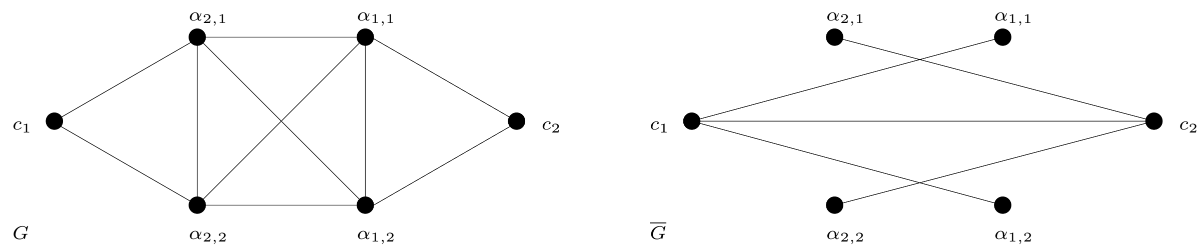

Figure 13. Graphs G and $\overline{G}$ in Example 5.2.

Theorem 5.3 (Jiang et al., 2006)**.** *Let $\overline{G}$ be the graph of $m \geq 3$ star graphs with centers $c_1, c_2, \dots, c_m$ where c_i is adjacent to $\beta_i \geq 1$ degree 1 vertices $\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,\beta_i}$ and all c_i are adjacent to each other. Then G is not the graph of a semigroup.*

5.2. **Removing 2 modified star graphs with shared vertices.**

Theorem 5.4. *Let $\overline{G}$ be the graph of 2 star graphs with centers c_1, c_2 where c_i is adjacent to $\beta_i \geq 2$ degree 1 vertices $\alpha_{i,1}, \dots, \alpha_{i,\beta_i}$; c_1, c_2 are adjacent; and c_1, c_2 are both adjacent to $\beta_3 \geq 1$ degree 2 vertices $\alpha_{3,1}, \dots, \alpha_{3,\beta_3}$. Then G is the graph of a semigroup.*

Proof. Let $S = V \cup \{0\}$ with operations $c_i^2 = c_i$, $c_1c_2 = \alpha_{3,1}$, $c_i\alpha_{i,j} = \alpha_{i,1}$, $c_i\alpha_{3,1} = \alpha_{3,1}$, $c_i\alpha_{3,j\neq 1} = \alpha_{i,1}$, $\alpha_{3,1}^2 = \alpha_{3,1}$, and all other products equal to 0. Checking associativity, $(c_ic_i)c_j = c_ic_j = \alpha_{3,1} = c_i\alpha_{3,1} = c_i(c_ic_j)$ for $i \neq j$. If $\alpha_{i,j} \neq \alpha_{3,1}$, then $(c_1c_2)\alpha_{i,j} = \alpha_{3,1}\alpha_{i,j} = 0 = c_1(c_2\alpha_{i,j}) = c_2(c_1\alpha_{i,j})$. Next, we have $(c_1c_2)\alpha_{3,1} = \alpha_{3,1} = c_1(c_2\alpha_{3,1})$. If $i \neq k, 3$, then $c_k(c_k\alpha_{i,j}) = 0 = (c_k^2)\alpha_{i,j}$. Otherwise, $c_k(c_k\alpha_{i,j}) = c_k\alpha_{i,1} = (c_k^2)\alpha_{i,j}$. If $\alpha_{i,j} = \alpha_{h,l} = \alpha_{3,1}$, then $c_k(\alpha_{3,1}\alpha_{3,1}) = \alpha_{3,1} = (c_k\alpha_{3,1})\alpha_{3,1}$. Otherwise $c_k(\alpha_{i,j}\alpha_{h,l}) = 0 = (c_k\alpha_{i,j})\alpha_{h,l}$. Finally, multiplying three α's will equal 0 unless all three are $\alpha_{3,1}$. Therefore S is a semigroup, and clearly $G = \Gamma(S)$. ■

6. Conclusion and open problems

Theorem 2.5 proves that all graphs with a sufficient number of edges are zero-divisor graphs. Corollary 2.3, Lemma 3.14, and Corollary 3.19 give conditions when one may determine if a graph is a zero-divisor graph by inspecting smaller subgraphs. The annihilator, open neighborhood, and closed neighborhood equivalence relations coupled with quotient graphs describe properties of the possible semigroups associated to a graph. While working with quotient graphs, we observed the following conjecture, which we leave as an open problem.

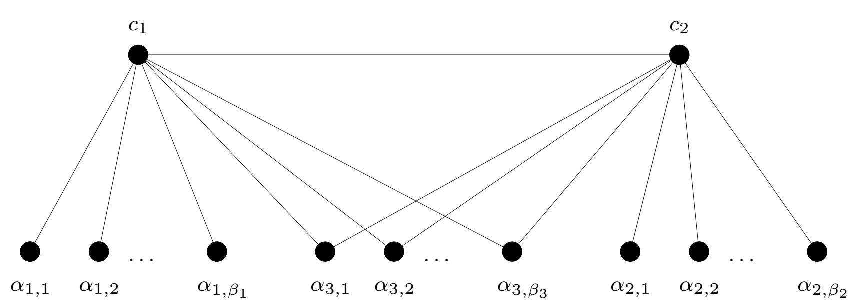

Figure 14. $\overline{G}$ satisfying conditions of Theorem 5.4.

Conjecture 6.1. *If G is a zero-divisor graph, then so are $\Xi_o(G)$ and $\Xi_c(G)$.*

Theorem 3.23 and Corollary 3.28 answer the question of when adding at least 3 degree 2 vertices to a complete graph results in a zero-divisor graph. The problem of adding higher degree vertices to a complete graph is still unsolved.

Theorem 4.4 provides a necessary and sufficient condition for determining regular zero-divisor graphs. From our study of maximum vertex degrees, we observe that:

Conjecture 6.2. *If $\Delta \leq \lceil |G|/3 \rceil$, then G is not the graph of a semigroup.*

We used a computer program to check that Conjecture 6.2 holds for all graphs with $|G| \leq 12$. The following example shows the difficulty in proving this conjecture.

Example 6.3. Construct a graph G with $V = \{a_{i,j} : 1 \leq i \leq 10, 1 \leq j \leq 4\} \cup \{x_i : 1 \leq i \leq 10\}$, $N(a_{i,j}) = \{x_i, x_{i+1 \pmod{10}}\}$, and $G[\{x_i\}] = K_{10}$. As $|G| = 50$ and $\Delta = 17$, we have $\Delta = \lceil |G|/3 \rceil$, which satisfies Conjecture 6.2. Suppose $G = \Gamma(S)$. Then by Proposition 3.1, removing all vertices $a_{i,j}$ for $j > 1$ results in the graph of a semigroup. This graph is, however, K_{10} with 10 triangles and not the refinement of a star graph, and therefore not the graph of a semigroup by Corollary 3.28. Hence G cannot be the graph of a semigroup.

The graph G, however, satisfies all previously found necessary conditions, indicating that Conjecture 6.2 cannot be proved using only known results.

Finally, Theorem 5.1 proves a general class of graphs are never zero-divisor graphs, including many graphs that satisfy all previous conditions. We are, however, still trying to find a necessary and sufficient condition for determining zero-divisor graphs.

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