A New Infinite Family of Minimally Nonideal Matrices

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Abstract

Minimally nonideal matrices are a key to understanding when the set covering problem can be solved using linear programming. The complete classification of minimally nonideal matrices is an open problem. One of the most important results on these matrices comes from a theorem of Lehman, which gives a property of the core of a minimally nonideal matrix. Cornuéjols and Novick gave a conjecture on the possible cores of minimally nonideal matrices. This paper disproves their conjecture by constructing a new infinite family of square minimally nonideal matrices. In particular, we show that there exists a minimally nonideal matrix with r ones in each row and column for any $r \geq 3$.

Key words: Minimally nonideal matrices, Ideal matrices, Lehman matrices, Set covering polyhedra
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1. Introduction

Minimally nonideal matrices are a key to understanding when the set covering problem problem can be solved using linear programming. The set covering problem is a fundamental problem in combinatorial optimization [1], and many combinatorial problems can be reduced to it. We can represent a collection of sets with a $0,1$ $m \times n$ matrix A by letting the rows be the elements to cover and the columns be the sets, with $a_{ij} = 1$ if set j contains element i. Then the set covering problem may be formulated as finding

$$\min\{c^T x \mid Ax \geq \mathbf{1}, x \in \{0,1\}^n\},$$

where $\mathbf{1}$ is the vector with ones in all entries, inequalities hold coordinate-wise, and $c \in \mathbb{R}^n$ is an objective function.

Define the *set covering polyhedron* of A by

$$Q(A) = \{x \in \mathbb{R}^n \mid Ax \geq \mathbf{1}, x \geq 0\}.$$

A point $x \in Q(A)$ is called an *extreme point* if for any two points $y, z \in Q(A)$ such that $x = (y+z)/2$, we must have $x = y = z$. A matrix A is *ideal* if $Q(A)$ is *integral*, i.e.,

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all its extreme points have integer coordinates. The set covering problem is NP-complete in general, but in the special case when A is ideal, the problem can be solved using linear programming for any objective function c. Ideal matrices are also known as *width-length matrices* [2, 3], matrices with the *weak max-flow min-cut property* [4], or matrices with the *max-flow min-cut property* [2, 3]. One natural way to study ideal matrices is to consider the "smallest" possible matrices that are not ideal. A matrix A is *minimally nonideal (mni)* if

1. A does not contain a dominating row,
2. A is not ideal, and
3. for all $i = 1, \ldots, n$, the two polyhedra $Q(A) \cap \{x \mid x_i = 0\}$ and $Q(A) \cap \{x \mid x_i = 1\}$ are integral.

A row x of A is *dominating* if for some other row y we have $x \geq y$.

Define the *circulant matrix* C_n^r as the $n \times n$ matrix with columns indexed by $\mathbb{Z}/n\mathbb{Z}$ and rows equal to the incidence vectors of $\{i, i+1, \ldots, i+r-1\}$ for $i \in \mathbb{Z}/n\mathbb{Z}$. Also define the point-line incidence matrix of a *degenerate projective plane* $\mathcal{J}_n$ for $n \geq 2$ to be the square $(n+1) \times (n+1)$ matrix with columns indexed by $\{0, \ldots, n\}$ and rows equal to the incidence vectors of $\{1, \ldots, n\}, \{0, 1\}, \{0, 2\}, \ldots, \{0, n\}$. Lehman [2] noted that C_n^2 for $n \geq 3$ odd and $\mathcal{J}_n$ for $n \geq 2$ are mni matrices.

Let the *blocker* $b(A)$ be the matrix with n columns where the rows are the minimal $0, 1$ vectors $x^T \in \{0, 1\}^n$ under the dominance ordering such that $x \in Q(A)$. A $0, 1$ matrix A is an mni matrix if and only if its blocker $b(A)$ is an mni matrix [2]. Lehman proved the following seminal theorem on the structure of mni matrices.

Theorem 1 ([3])**.** *If A is an mni matrix, then $Q(A)$ has a unique fractional extreme point, and either*

1. *A is isomorphic to $\mathcal{J}_n$ for $n \geq 2$, or*
2. *the rows of A (resp. $b(A)$) may be permuted so that A (resp. $b(A)$) contains a square $n \times n$ submatrix $\bar{A}$ (resp. $\bar{B}$) with exactly $r \geq 2$ (resp. $s \geq 2$) ones in each row and column, and $\bar{A}\bar{B}^T = J_n + (rs - n)I_n$ (where J_n is the matrix of all ones). Moreover, every other row of A (resp. $b(A)$) has more than r (resp. s) ones.*

Two matrices are *isomorphic* if one can be obtained from the other by permutations of rows and/or columns. The square submatrix $\bar{A}$ is called the *core* of A. The core of A is unique up to isomorphism.

Cornuéjols and Novick [5] characterized all ideal and mni circulant matrices C_n^r. They also conjectured that C_n^2 and $C_n^{(n+1)/2}$ for $n \geq 3$ odd are essentially the only possible cores of mni matrices.

Conjecture 2 ([5], Conjecture 1.2)**.** *There exists an n_0 such that except for the degenerate projective planes $\mathcal{J}_n$, each mni matrix with $n \geq n_0$ has core isomorphic to C_n^2 or $C_n^{(n+1)/2}$ for $n \geq 3$ odd.*

In order to study cores of mni matrices, Lütolf and Margot [6] defined the class of *Lehman matrices*. Two square $0,1$ $n \times n$ matrices A, B form a pair of Lehman matrices if $AB^T = J_n + dI_n$ for some positive integer d. Bridges and Ryser [7] showed that a Lehman matrix must have the same number of ones in each row and column. Two infinite families of Lehman matrices are known: point-line incidence matrices of nondegenerate finite projective planes [8] and Lehman matrices with $d = 1$. Novick [9] showed that the only nondegenerate finite projective plane with an mni point-line incidence matrix is the Fano plane F_7. Cornuéjols et al. [10] studied Lehman matrices with $d = 1$ according to their similarity to the circulant matrices C_n^r. Wang [11] used graphs to provide additional properties of Lehman matrices with $d = 1$.

In this paper, we describe a new infinite family of square mni matrices and disprove Conjecture 2. In Section 2, for each $r \geq 3$ we construct a $0,1$ $(r^2-1) \times (r^2-1)$ matrix Ω_r. We show in Section 3 that Ω_r is a Lehman matrix with $d = 1$, i.e., there exists for each r a square $0,1$ matrix B such that $\Omega_r B^T = J_{r^2-1} + I_{r^2-1}$. In Section 4, we prove that Ω_r is mni by showing $Q(\Omega_r)$ has a unique fractional extreme point. We also describe the blocker of Ω_r. Our results show that there exists an mni matrix with r ones in each row and column for any $r \geq 3$.

2. Construction

Let $r \geq 3$ and $n = r^2 - 1$. Let J_{r-1} denote the $(r-1) \times (r-1)$ matrix with all entries equal to 1, and let E_{ij} be the $0,1$ $(r-1) \times (r-1)$ matrix with a single 1 in row i, column j. We define the $n \times n$ matrix Ω_r by

$$\Omega_r := \begin{bmatrix} J_{r-1} & E_{11} & E_{22} & \dots & E_{r-1,r-1} & 0 \\ 0 & J_{r-1} & E_{11} & & E_{r-2,r-2} & E_{r-1,r-1} \\ E_{r-1,r-1} & 0 & J_{r-1} & & E_{r-3,r-3} & E_{r-2,r-2} \\ \vdots & & & \ddots & & \vdots \\ E_{22} & E_{33} & E_{44} & & J_{r-1} & E_{11} \\ E_{11} & E_{22} & E_{33} & \dots & 0 & J_{r-1} \end{bmatrix},$$

where each block is $(r-1) \times (r-1)$.

Recall that if X is an $m \times n$ matrix and Y is a $k \times \ell$ matrix, then the Kronecker product $X \otimes Y$ is the $mk \times n\ell$ block matrix

$$X \otimes Y := \begin{bmatrix} x_{11}Y & \dots & x_{1n}Y \\ \vdots & & \vdots \\ x_{m1}Y & \dots & x_{mn}Y \end{bmatrix}.$$

If we let P be the $(r+1)\times(r+1)$ matrix

$$P := \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{bmatrix},$$

then Ω_r can be more succinctly expressed using Kronecker products as

$$\Omega_r = I_{r+1} \otimes J_{r-1} + P \otimes E_{11} + P^2 \otimes E_{22} + \cdots + P^{r-1} \otimes E_{r-1,r-1}.$$

Our goal is to prove the following theorem.

Theorem 3. *For any $r \geq 3$, the matrix Ω_r is an mni matrix.*

3. Ω_r is Lehman

We first introduce some new notation that will be used throughout the rest of this paper. For $i \in \{1, \ldots, r+1\}$ and $j \in \{1, \ldots, r-1\}$, define

$$i \otimes j := (i-1)(r-1) + j \in \{1, \ldots, n\}.$$

The motivation for this notation is that we can decompose a vector $x \in \mathbb{R}^n$ into

$$x = \sum_{i=1}^{r+1} \sum_{j=1}^{r-1} x_{i\otimes j}(u_i \otimes v_j), \quad x_{i\otimes j} \in \mathbb{R}$$

where u_i, v_j are the standard basis vectors for $\mathbb{R}^{r+1}, \mathbb{R}^{r-1}$.

Let $\pi \in S_{r+1}$ be the permutation of order $r+1$ defined by

$$\pi(i) = i + 1 \text{ for } i \leq r, \quad \pi(r+1) = 1.$$

Define $A_{i\otimes j}$ to be the $(i\otimes j)^{\text{th}}$ row vector of the matrix Ω_r. By definition, $A_{i\otimes j}$ is the incidence vector of

$$\{i \otimes 1, \ldots, i \otimes (r-1), \pi^j(i) \otimes j\}. \tag{1}$$

Set the $0,1$ vector $y^{i\otimes j}$ to be the incidence vector of

$$\{\pi(i) \otimes 1, \ldots, \pi^{r-1}(i) \otimes (r-1), \pi^r(i) \otimes j\}. \tag{2}$$

We use the above notation to prove that Ω_r is a Lehman matrix with $d = 1$.

Lemma 4. *There exists a $0,1$ $n \times n$ matrix B satisfying*

$$\Omega_r B^T = J_n + I_n.$$

Proof. Let $i \in \{1, \dots, r+1\}$ and $j \in \{1, \dots, r-1\}$. Since $\pi^0(i) = i$, we see from (1) that

$$\begin{aligned} A_{i\otimes\ell} y^{i\otimes j} &= 1, & \ell &\in \{1, \dots, r-1\} \\ A_{\pi^r(i)\otimes\ell} y^{i\otimes j} &= 1, & \ell &\in \{1, \dots, r-1\} \\ A_{\pi^k(i)\otimes\ell} y^{i\otimes j} &= 1 + \delta_{k,r-j}\delta_{j\ell}, & k, \ell &\in \{1, \dots, r-1\}. \end{aligned} \tag{3}$$

Therefore $\Omega_r y^{i\otimes j} = \mathbf{1} + u_{\pi^{r-j}(i)} \otimes v_j$. If we define B to be the $n \times n$ matrix with row $i \otimes j$ equal to $y^{\pi^{j+1}(i)\otimes j}$, then

$$\Omega_r B^T = J_n + I_n.$$

□

4. Proof of Theorem 3

In order to prove Ω_r is mni, we must study the set covering polyhedron

$$Q(\Omega_r) = \{x \in \mathbb{R}^n \mid \Omega_r x \geq \mathbf{1}, x \geq 0\}.$$

Recall that a point $x \in Q(\Omega_r)$ is called an extreme point if for any two points $y, z \in Q(\Omega_r)$ such that $x = (y+z)/2$, we must have $x = y = z$. An easy extension of the definition shows that if x is an extreme point and x dominates a convex combination of points of $Q(\Omega_r)$, then x must be equal to one of the points in the convex combination. Note that an extreme point satisfies $x \leq \mathbf{1}$ since Ω_r is a $0,1$ matrix.

The following two lemmas by Lütolf and Margot [6] provide sufficient conditions for a Lehman matrix to be mni.

Lemma 5 ([6], Lemma 2.7)**.** *If A is a Lehman matrix with r ones in each row and column, then $(1/r, \dots, 1/r)$ is a fractional extreme point of $Q(A)$.*

Lemma 6 ([6], Lemma 2.8)**.** *If A is a Lehman matrix such that $Q(A)$ has a unique fractional extreme point, then A is an mni matrix.*

We see from the construction that Ω_r has r ones in each row and column. It follows from the previous two lemmas that the following theorem implies Theorem 3.

Theorem 7. *For $r \geq 3$, the polyhedron $Q(\Omega_r)$ has unique fractional extreme point $(1/r, \dots, 1/r)$.*

We use the notation introduced in Section 3. Define the linear map $\eta_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, \dots, r+1\}$ by

$$\eta_i(x) := \sum_{j=1}^{r-1} x_{i\otimes j}.$$

Let Z denote the set of n-dimensional $0,1$ vectors that contains exactly one 1 in each $(r-1)$ block, so

$$Z = \{z \in \{0,1\}^n \mid \eta_i(z) = 1 \text{ for all } i = 1, \dots, r+1\}.$$

Observe that $Z \subset Q(\Omega_r)$.

The maps η_i play a central role in the proof of Theorem 7. We first present two lemmas related to these maps.

Lemma 8. *If $x \in \mathbb{R}^n, x \geq 0$ satisfies $\eta_i(x) \geq \alpha > 0$ for all $i = 1, \ldots, r+1$, then*

$$x \geq \sum_{z \in Z} \lambda_z z$$

for some multipliers $\lambda_z \in \mathbb{R}_{\geq 0}$ satisfying $\sum \lambda_z = \alpha$.

Proof. The proof is by induction on the number of nonzero coordinates of the vector x. Since $\alpha > 0$, there is some index t such that x_t is the smallest nonzero coordinate of x. As $\eta_i(x) > 0$ for all i, there must exist $z^* \in Z$ such that $z^*_t = 1$ and $x_t z^* \leq x$. Then $x^* = x - x_t z^* \geq 0$. If $x_t \geq \alpha$, the claimed result holds because

$$x \geq \alpha z^*.$$

If $x_t < \alpha$, then $\eta_i(x^*) \geq \alpha - x_t > 0$ for all i, and x^* has one more zero coordinate than x. By induction we have $x^* \geq \sum_{z \in Z} \lambda^*_z z$ for some multipliers $\lambda^*_z \in \mathbb{R}_{\geq 0}$ satisfying $\sum \lambda^*_z = \alpha - x_t$. Now the claimed result follows by choosing

$$\lambda_z = \begin{cases} \lambda^*_z & \text{if } z \neq z^* \\ \lambda^*_z + x_t & \text{if } z = z^* \end{cases}.$$ □

We will henceforth use η_i to denote $\eta_i(x)$ when there is no ambiguity.

Lemma 9. *For $x \in Q(\Omega_r)$ and $i \in \{1, \ldots, r+1\}$,*

$$\eta_1 + \cdots + \eta_{r+1} \geq \eta_i - \max(\eta_i, 1) - \max(\eta_{\pi^{-1}(i)}, 1) + \sum_{k=1}^{r+1} \max(\eta_k, 1).$$

Proof. From (1), we see that

$$A_{\ell \otimes j} x = \eta_\ell + x_{\pi^j(\ell) \otimes j}. \tag{4}$$

Consider the row $\pi^i(r-j) \otimes j$ of Ω_r for $j \in \{1, \ldots, r-1\}$. By (4) and $x \in Q(\Omega_r)$,

$$A_{\pi^i(r-j) \otimes j} x = \eta_{\pi^i(r-j)} + x_{\pi^i(r) \otimes j} \geq \max(\eta_{\pi^i(r-j)}, 1).$$

Summing over all j gives

$$\eta_{\pi^i(r)} + \eta_{\pi^i(r-1)} + \cdots + \eta_{\pi^i(1)} \geq \max(\eta_{\pi^i(r-1)}, 1) + \cdots + \max(\eta_{\pi^i(1)}, 1).$$

As $\pi^i(r+1) = i$ and $\pi^i(r) = \pi^r(i) = \pi^{-1}(i)$, the LHS does not include η_i, and the RHS does not include $\max(\eta_i, 1)$ or $\max(\eta_{\pi^{-1}(i)}, 1)$. Adding η_i to both sides of the inequality, we get the claim. □

We will now use the previous two lemmas to prove Theorem 7.

Proof of Theorem 7. Let x be an extreme point of $Q(\Omega_r)$ not equal to $(1/r, \ldots, 1/r)$. We split the problem into two cases. We show that if all $\eta_i \geq 1$, then x must be an element of the set Z. If there exists $i \in \{1, \ldots, r+1\}$ such that $\eta_i < 1$, then we prove $x = y^{i \otimes j}$ for some $j \in \{1, \ldots, r-1\}$, where $y^{i \otimes j}$ is defined in (2). Since x must be integral in both cases, $(1/r, \ldots, 1/r)$ is the only fractional extreme point.

Case 1. Suppose $\eta_i \geq 1$ for all i.

By a direct application of Lemma 8, we have for some multipliers $\lambda_z \in \mathbb{R}_{\geq 0}$,

$$x \geq \sum_{z \in Z} \lambda_z z, \quad \sum \lambda_z = 1.$$

Since x is an extreme point and $\sum \lambda_z z$ is a convex combination of points in Z, we must have $x \in Z$.

Case 2. Suppose $\min \eta_i < 1$.

We will prove that in this case $x \in \{y^{i \otimes j}\}$. Let k be such that $\eta_k = \min \eta_i < 1$. We first show that we may assume $k = 1$. Define the $n \times n$ permutation matrix $M = P^{k-1} \otimes I_{r-1}$. Observe that because I_{r+1}, P^i are conjugation invariant under P^{k-1},

$$M \Omega_r M^{-1} = (P^{k-1} \otimes I_{r-1}) \Omega_r (P^{-k+1} \otimes I_{r-1}) = \Omega_r.$$

Note that since M is a permutation matrix, Mx is an extreme point of

$$\begin{aligned} M(Q(\Omega_r)) &= \{Mx' \mid \Omega_r x' = \Omega_r M^{-1} M x' \geq \mathbf{1}, x' \geq 0\} \\ &= \{x'' \mid \Omega_r M^{-1} x'' \geq \mathbf{1}, x'' \geq 0\} \\ &= Q(M \Omega_r M^{-1}) = Q(\Omega_r). \end{aligned}$$

Now if $Mx = y^{i \otimes j}$ for some $i \in \{1, \ldots, r+1\}, j \in \{1, \ldots, r-1\}$, then

$$x = M^{-1} y^{i \otimes j} = (P^{-k+1} \otimes I_{r-1}) y^{i \otimes j} = y^{\pi^{k-1}(i) \otimes j}.$$

Since $\eta_\ell(Mx) = \eta_\ell((P^{k-1} \otimes I_{r-1})x) = \eta_{\pi^{k-1}(\ell)}(x)$ for all ℓ, we may replace x with Mx and assume that $\eta_1 = \min \eta_\ell$.

Claim 1. *There exists $j \in \{1, \ldots, r-1\}$ such that $x_{(r+1) \otimes j} > 0$ and $A_{\pi^{-j}(r+1) \otimes j} x > 1$.*

Suppose for the sake of contradiction that for each $j \in \{1, \ldots, r-1\}$, either

$$x_{(r+1) \otimes j} = 0 \quad \text{or} \quad A_{\pi^{-j}(r+1) \otimes j} x \leq 1.$$

If $x_{(r+1) \otimes j} = 0$, then since $x \in Q(\Omega_r)$, we have from (4) that

$$A_{\pi^{-j}(r+1) \otimes j} x = \eta_{\pi^{-j}(r+1)} \geq 1 \implies A_{\pi^{-j}(r+1) \otimes j} x = \max(\eta_{\pi^{-j}(r+1)}, 1).$$

We deduce that for every j,

$$\eta_{\pi^{-j}(r+1)} + x_{(r+1) \otimes j} = A_{\pi^{-j}(r+1) \otimes j} x \leq \max(\eta_{\pi^{-j}(r+1)}, 1).$$

Summing over all j and adding η_1, we conclude that

$$\eta_1 + \cdots + \eta_{r+1} = \eta_1 + \sum_{j=1}^{r-1} A_{\pi^{-j}(r+1)\otimes j}x \leq \eta_1 + \sum_{k=2}^{r} \max(\eta_k, 1). \tag{5}$$

We prove that $\eta_1 = \eta_i$ for all $i \in \{1, \ldots, r+1\}$ by induction on i. The claim is clear for $i = 1$. Take $i > 1$ and suppose $\eta_1 = \eta_{i-1} < 1$. Then combining Lemma 9 and (5),

$$\eta_1 + \sum_{k=2}^{r} \max(\eta_k, 1) \geq \sum_{k=1}^{r+1} \eta_k \geq \eta_i - \max(\eta_i, 1) - \max(\eta_{\pi^{-1}(i)}, 1) + \sum_{k=1}^{r+1} \max(\eta_k, 1).$$

Since $\pi^{-1}(i) = i - 1$, we observe that $\max(\eta_1, 1) - \max(\eta_{\pi^{-1}(i)}, 1) = 1 - 1 = 0$. Canceling terms, we have

$$\eta_1 \geq \eta_i - \max(\eta_i, 1) + \max(\eta_{r+1}, 1) \geq \eta_i + 1 - \max(\eta_i, 1) \geq \min(\eta_i, 1).$$

Since $\eta_1 < 1$ and we chose η_1 to be minimal, we have $\eta_1 = \eta_i$. This concludes the inductive step. Therefore

$$\eta_1 = \cdots = \eta_{r+1} < 1. \tag{6}$$

It is a fact of convexity theory that if $x \in \mathbb{R}^n$ is an extreme point of the polyhedron defined by the system of inequalities $\Omega_r x' \geq \mathbf{1}, x' \geq 0$, then x must satisfy at least n of the inequalities with equalities [12]. From Lemma 4, we know that there exists B such that $\Omega_r B^T = J_n + I_n$. Since Ω_r has r ones in each row,

$$\Omega_r(B^T - \tfrac{1}{r}J_n) = I_n,$$

so Ω_r is invertible. Thus the point $(1/r, \ldots, 1/r)$ is the unique solution to $\Omega_r x = \mathbf{1}$. Therefore $x \neq (1/r, \ldots, 1/r)$ must satisfy $x_{i\otimes j} = 0$ for some i, j. As

$$A_{\pi^{-j}(i)\otimes j}x = \eta_{\pi^{-j}(i)} + x_{i\otimes j} = \eta_{\pi^{-j}(i)} \geq 1,$$

we have a contradiction of (6). This proves the claim.

Take $j \in \{1, \ldots, r-1\}$ from Claim 1 such that $x_{(r+1)\otimes j} > 0$ and $A_{\pi^{-j}(r+1)\otimes j}x > 1$. We show that there exists $\epsilon_0 \in \mathbb{R}_{>0}$ such that $\epsilon_0 y^{1\otimes j} \leq x$ (i.e., x is nonzero in every coordinate where $y^{1\otimes j}$ is nonzero). For $\ell \in \{1, \ldots, r-1\}$,

$$A_{1\otimes \ell}x = \eta_1 + x_{(\ell+1)\otimes \ell} \geq 1$$

implies $x_{(\ell+1)\otimes \ell} \geq 1 - \eta_1 > 0$. From the definition of $y^{1\otimes j}$, it suffices to take

$$\epsilon_0 = \min(x_{2\otimes 1}, x_{3\otimes 2}, \ldots, x_{r\otimes(r-1)}, x_{(r+1)\otimes j}) > 0.$$

We claim that x must in fact be equal to $y^{1\otimes j}$ since x is an extreme point. Let $y = y^{1\otimes j}$. For any $0 < \epsilon < 1$, we have $(1-\epsilon)x + \epsilon y \in Q(\Omega_r)$ by convexity from $x, y \in Q(\Omega_r)$. For any

$0 < \epsilon < \epsilon_0$, we have $(1+\epsilon)x - \epsilon y \geq x - \epsilon y \geq 0$. For any row $k \otimes \ell$ not equal to $\pi^{-j}(r+1) \otimes j$, we have

$$A_{k\otimes\ell}((1+\epsilon)x - \epsilon y) = (1+\epsilon)A_{k\otimes\ell}x - \epsilon \geq 1.$$

If we additionally assume $0 < \epsilon < (A_{\pi^{-j}(r+1)\otimes j}x - 1)/2$, then

$$A_{\pi^{-j}(r+1)\otimes j}((1+\epsilon)x - \epsilon y) = (1+\epsilon)A_{\pi^{-j}(r+1)\otimes j}x - 2\epsilon \geq A_{\pi^{-j}(r+1)\otimes j}x - 2\epsilon \geq 1.$$

We conclude that for any $0 < \epsilon < \min(1, \epsilon_0, (A_{\pi^{-j}(r+1)\otimes j}x - 1)/2)$, the two points

$$(1 \mp \epsilon)x \pm \epsilon y \in Q(\Omega_r).$$

Since x is an extreme point, this implies $(1-\epsilon)x + \epsilon y = (1+\epsilon)x - \epsilon y$, which is equivalent to $x = y$.

Therefore the only fractional extreme point of $Q(\Omega_r)$ is the point $(1/r, \ldots, 1/r)$. □

Note that in the course of proving Theorem 7, we showed that the $0, 1$ extreme points of $Q(\Omega_r)$ are a subset of $Z \cup \{y^{i\otimes j}\}$. Recall that the blocker $b(\Omega_r)$ is the matrix with rows corresponding to the minimal $0, 1$ vectors x^T such that $x \in Q(\Omega_r)$, or equivalently the $0, 1$ extreme points of $Q(\Omega_r)$ [1, Remark 1.16]. There are $(r+1)(r-1) = r^2 - 1$ distinct $y^{i\otimes j}$. The set Z has $(r-1)^{r+1}$ elements, but exactly $(r-1)$ distinct vectors in Z dominate each $y^{i\otimes j}$. Therefore $b(\Omega_r)$ has $(r-1)^{r+1} - (r-2)(r^2-1)$ rows, and they are exactly the minimal vectors in $Z \cup \{y^{i\otimes j}\}$.

Lemma 6 shows that Theorem 7 implies Theorem 3. Since Ω_r is a square matrix, Theorem 1 implies that it is equal to its core. Rows $1, \ldots, r-1$ of Ω_r have $r-1$ ones in the same $r-1$ columns, and the number of shared columns is invariant under isomorphism, so Ω_r is not isomorphic to C_n^r. Lastly Ω_r has r ones in every row, so it cannot be isomorphic to $\mathcal{J}_s$. Thus we have a new infinite family of mni matrices and cores, and Conjecture 2 is disproved. Theorem 3 also shows that the core of an mni matrix can contain exactly r ones in every row and column, for any $r \geq 3$.

5. Open problems

The complete classification of mni matrices is still an open problem. In fact, it is unknown which square Lehman matrices are cores of mni matrices. We have shown that there are more possible cores than previously proposed in Conjecture 2. Matrices other than those in the conjecture and the newly constructed Ω_r may, however, still be cores of mni matrices. Many Lehman matrices are not mni, but we were unable to prove that there exists a Lehman matrix with $d = 1$ that is not the core of an mni matrix. We believe that such a matrix should exist, and such an example would be very useful.

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