# INTRODUCTION TO $\mathcal{D}$-MODULES AND REPRESENTATION THEORY

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## 1. Introduction

The goal of this exposition is to give a streamlined introduction to the basic theory of $\mathcal{D}$-modules and then use them to give an overview of some preliminaries in geometric representation theory.

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1.1. **Background.** We use algebraic geometry heavily throughout, citing EGA and [Har77] when appropriate. We will also need to consider sheaves of rings and modules in the non-quasi-coherent setting as well. For this, we found [KS94] to be excellent. The derived category is essential in working with functors of $\mathcal{D}$-modules. The book of [GM03] seems to be the most comprehensive (see also [Wei94], [KS94], and [HTT08, Appendix]). The language of stacks provides a helpful way of conceptualizing some of the ideas we present. For an introduction to fibered categories, stacks, and descent, see [FGI$^+$05, Chapter 1]. For algebraic stacks, we will only need what is presented in [Wan11], but more general references include [sta] and [LMB00].

On the representation theory side, [DG70] is the canonical reference for generalities on affine group schemes. Newer references in English include [Mil12], [Jan03], and [Wat79].

1.2. **Outline.** This essay consists of two parts. Part I, consisting of §2-4, develops the general theory of calculus in the algebraic setting. We start by defining twisted differential operators, with the main result showing that these objects satisfy smooth descent and hence can be defined on an algebraic stack. The material largely follows [BB93]. The heart of the essay lies in §3-4, where we have tried to give a succinct yet thorough introduction to the theory of $\mathcal{D}$-modules on smooth quasi-projective schemes. The organization and material follows the fantastic lecture notes of [Ber84], though we appeal to [BGK$^+$87] and [HTT08] to fill in some details. Our approach in terms of proofs and references here was to balance readability and detailedness. We only cite proofs that we found to be very easily accessible. In §5, the second part of the essay, we develop the framework of equivariant objects using stacks. The goal here is to set up a versatile theory that can then be applied to the case of flag varieties. The idea we had in mind was to use the equivalence $G\backslash(G/H) \simeq (\cdot/H)$ of quotient stacks to relate $G$-equivariant objects on the quotient space $G/H$ with $H$-actions on vector spaces. The main reference for this section was again [BB93]. The expositions of [Gai05], [Kas89], and [Sun11] were also immensely helpful overall guides during the writing process.

1.3. **Notation.** We fix an algebraically closed field $k$ of characteristic 0, and we will work in the category $\mathbf{Sch}_{/k}$ of schemes over $k$. All schemes will be $k$-schemes and all sheaves will be of $k$-vector spaces. For two $k$-schemes $X$ and $Y$, we will use $X \times Y$ to denote $X \times_{\mathrm{Spec}\, k} Y$. Tensor products will be taken over $k$ if not specified. We will consider a scheme as both a geometric object and a sheaf on the site $(\mathbf{Sch})_{\mathrm{fpqc}}$ without distinguishing the two. For $k$-schemes $X$ and $S$, we denote $X(S) = \mathrm{Hom}_k(S, X)$. For two morphisms of schemes $X \to S$ and $Y \to S$, we use $\mathrm{pr}_1 : X \times_S Y \to X$ and $\mathrm{pr}_2 : X \times_S Y \to Y$ to denote the projection morphisms when there is no ambiguity. For a point $s \in S$, we let $\kappa(s)$ equal the residue field.

For a scheme $X$, we let $\Omega^1_X$ denote the sheaf of differential 1-forms. We use $\mathcal{H}om, \mathcal{D}er, \mathcal{E}nd$ to denote the corresponding sheaves. The tangent sheaf is defined as $\Theta_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$. For an open subset $U \subset X$, we will use $\mathcal{M}(U) = \Gamma(U, \mathcal{M})$ to denote local sections. When there is no confusion, we drop the subscript for $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$. We write $m \in \mathcal{F}$ to mean a local section in $\Gamma(U, \mathcal{F})$ for some open $U$. We will think of vector bundles both as locally free sheaves and as geometric schemes. Let $\varphi : Y \to X$ be a morphism of schemes. We let $f^{-1}$ denote the sheaf inverse image and $f^*$ the inverse image of $\mathcal{O}$-modules. The latter will also denote the same functor for quasi-coherent sheaves. To minimize confusion, we use $\varphi_\bullet$ to denote the sheaf direct image, which preserves quasi-coherence when $\varphi$ is quasi-compact.

If $\mathcal{M}, \mathcal{N}$ are sheaves on schemes $X, Y$ respectively, we use $\mathcal{M} \boxtimes_k \mathcal{N}$ to denote the sheaf $\mathrm{pr}_1^{-1}\mathcal{M} \otimes_k \mathrm{pr}_2^{-1}\mathcal{N}$ on $X \times Y$. If $\mathcal{M}, \mathcal{N}$ are $\mathcal{O}$-modules, then we use $\mathcal{M} \boxtimes_{\mathcal{O}} \mathcal{N}$ to denote

$$\mathcal{O}_{X \times Y} \underset{\mathcal{O}_X \boxtimes_k \mathcal{O}_Y}{\otimes} (\mathcal{M} \underset{k}{\boxtimes} \mathcal{N}).$$

In general, $\mathcal{O}_{X\times Y}$ does *not* equal $\mathcal{O}_X \boxtimes_k \mathcal{O}_Y$ because $X \times Y$ does not have the product topology. For $(x, y) \in X \times Y$, the stalk $\mathcal{O}_{X\times Y,(x,y)}$ is a localization of $\mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y}$, so we do have that $\mathcal{O}_{X\times Y}$ is flat over $\mathcal{O}_X \boxtimes \mathcal{O}_Y$.

For a sheaf of rings $\mathcal{A}$ on a scheme $X$, we let $\operatorname{Mod}(\mathcal{A})$ denote the abelian category of sheaves of $\mathcal{A}$-modules and $D(\mathcal{A})$ the derived category $D(\operatorname{Mod}(\mathcal{A}))$.

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## 2. Twisted differential operators

For a much more general and comprehensive account of differential modules, differential algebras, and twisted differential operators, see [BB93, §1-2]. We merely present the parts of their paper that we need and fill in some details. We also referred to [GD67, §16] and [Kas89].

### 2.1. Preliminary notions.

2.1.1. Let $X$ be a scheme locally of finite type over $k$, and let $X_\Delta^{(n)} \hookrightarrow X \times X$ be the $n$-th infinitesimal neighborhood of the diagonal. Explicitly, the diagonal $X \subset X \times X$ is covered by $U \times U$ for affine opens $U \subset X$. For $U = \operatorname{Spec} A$, let $I$ be the ideal defining the diagonal (i.e., the kernel of the multiplication map $A \otimes A \to A$), which is generated by $a \otimes 1 - 1 \otimes a$ for $a \in A$. Then $X_\Delta^{(n)} \cap U \times U = U_\Delta^{(n)} = \operatorname{Spec}(A \otimes A/I^{n+1})$.

Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-bimodules on $X$, which is quasi-coherent with respect to the left action. Then $M = \mathcal{M}(U)$ is an $(A \otimes A)$-module. We define a filtration $F_\bullet M$ on $M$ by $F_{-1}M = 0$, $F_n M = \operatorname{Hom}_{A\otimes A}(A \otimes A/I^{n+1}, M)$. Equivalently, $m \in F_n M$ if $(\operatorname{ad} a_0)\cdots(\operatorname{ad} a_n)m = 0$ for any $(n+1)$-tuple of elements of $A$, where $(\operatorname{ad} a)m = am - ma$. We say that $M$ is a *differential bimodule* if $M = \cup F_n M$. For $f \in A$, the opens $U_f \times U$, $U_f \times U_f$, and $U \times U_f$ coincide on $U_\Delta^{(n)}$. Considering $F_n M$ as a quasi-coherent sheaf on $U_\Delta^{(n)}$ shows that

$$A_f \underset{A}{\otimes} F_n M \simeq A_f \underset{A}{\otimes} F_n M \underset{A}{\otimes} A_f \simeq F_n M \underset{A}{\otimes} A_f.$$

Assuming $M$ is a differential bimodule, the above isomorphisms hold with $F_n M$ replaced by $M$. In particular, $\mathcal{M}(U_f) \simeq A_f \otimes_A M \simeq A_f \otimes_A M \otimes_A A_f$. Since $A$ is a finitely generated $k$-algebra, $I \subset A \otimes A$ is finitely generated, so $(A_f \otimes A_f) \otimes_{A\otimes A} F_n M \simeq F_n \mathcal{M}(U_f)$.

We say that $\mathcal{M}$ is a *differential* $\mathcal{O}_X$*-bimodule* if there is a covering of $X$ by open affines $U$ such that $\mathcal{M}(U)$ are differential bimodules. In this case, the above discussion shows that $\mathcal{M}$ is also quasi-coherent with respect to the right action, and $F_\bullet\mathcal{M}$ is a filtration of quasi-coherent $\mathcal{O}_X$-bimodules. Note that $F_n\mathcal{M}$ can be considered as a quasi-coherent sheaf on $X_\Delta^{(n)}$, with the two actions gotten via $\operatorname{pr}_{i*}^{(n)} : X_\Delta^{(n)} \hookrightarrow X \times X \rightrightarrows X$. If $X$ is quasi-separated, then $F_n\mathcal{M}$ is also quasi-coherent as a sheaf on the diagonal $X \times X$ for all $n$, so we can consider $\mathcal{M}$ as a quasi-coherent $\mathcal{O}_{X\times X}$-module.

2.1.2. Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. A $k$-linear morphism $P : \mathcal{F} \to \mathcal{G}$ is called a *differential operator* of order $\leq n$ if $P_U \in F_n \operatorname{Hom}_k(\mathcal{F}(U), \mathcal{G}(U))$ for any affine open $U \subset X$. Let $\mathcal{D}if^n(\mathcal{F}, \mathcal{G}) \subset \mathcal{H}om_k(\mathcal{F}, \mathcal{G})$ be the subsheaf of differential operators of order $\leq n$, which is naturally a $\mathcal{O}_X$-bimodule.

**Proposition 2.1.3.** *If $\mathcal{F}$ is coherent, then $\mathcal{D}if^n(\mathcal{F}, \mathcal{G})$ is a differential $\mathcal{O}_X$-bimodule.*

*Proof.* Cf. [GD67, Proposition 16.8.6]. It suffices to check quasi-coherence. Take an affine open $U = \operatorname{Spec} A \subset X$ and let $M = \mathcal{F}(U)$ and $N = \mathcal{G}(U)$. Let $\operatorname{Dif}^n(M,N) = F_n\operatorname{Hom}_k(M,N)$. Observe that

$$\operatorname{Dif}^n(M,N) = \operatorname{Hom}_{A\otimes A}(A\otimes A/I^{n+1}, \operatorname{Hom}_k(M,N)) \simeq \operatorname{Hom}_A(A\otimes A/I^{n+1} \underset{A}{\otimes} M, N)$$

where the tensor is with respect to the right $A$-action on $A\otimes A/I^{n+1}$ and the Hom is with respect to the left. Since $A$ is a finitely generated algebra, $A\otimes A/I^{n+1}$ is a finitely presented left $A$-module [GD67, Corollaire 16.4.22]. Therefore localization gives $A_f \otimes_A \operatorname{Dif}^n(M,N) \simeq \operatorname{Dif}^n(M_f, N_f)$. As a consequence, we get that $\mathcal{D}if^n(\mathcal{F},\mathcal{G})|_U \simeq \operatorname{Loc}_A \operatorname{Dif}^n(M,N)$. $\square$

Define the differential $\mathcal{O}_X$-bimodule $\mathcal{P}^n_X = (\mathrm{pr}_1^{(n)})_\bullet(\mathcal{O}_{X^{(n)}_\Delta})$. In general, this is locally finitely presented as an $\mathcal{O}_X$-module when $X$ is locally of finite presentation over $k$. The proof of Proposition 2.1.3 shows that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathcal{D}if^n(\mathcal{F},\mathcal{G}),$$

and we will often make use of this fact later to move tensor products around.

Let $\mathcal{D}if(\mathcal{F},\mathcal{G}) = \cup \mathcal{D}if^n(\mathcal{F},\mathcal{G})$. For coherent $\mathcal{F}$, this is a differential $\mathcal{O}_X$-bimodule.

2.1.4. We say that a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ is an $\mathcal{O}_X$*-differential algebra* if multiplication makes $\mathcal{A}$ a differential $\mathcal{O}_X$-bimodule. Then the commutator $[a,b] = ab - ba$ satisfies the identities $[a,bc] = [a,b]c + b[a,c]$ and $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ for $a,b,c \in \mathcal{A}$. Induction shows that $F_\bullet\mathcal{A}$ is a ring filtration and the associated graded algebra $\operatorname{gr}^F \mathcal{A}$ is commutative.

**Example 2.1.5.** (i) For coherent $\mathcal{F}$, the differential operators $\mathcal{D}_\mathcal{F} := \mathcal{D}if(\mathcal{F},\mathcal{F})$ form a differential algebra. Put $\mathcal{D}_X = \mathcal{D}_{\mathcal{O}_X}$. This is the algebra we will be focusing on.

(ii) Suppose $\mathcal{A}$, $\mathcal{B}$ are differential algebras on schemes $X, Y$ respectively. Then $\mathcal{A} \boxtimes_k \mathcal{B}$ is an $\mathcal{O}_X \boxtimes_k \mathcal{O}_Y$-algebra. Define $\mathcal{A}\boxtimes_\mathcal{O} \mathcal{B}$ with respect to the left multiplication. Then for open affines $U \subset X$, $V \subset Y$, we have $(\mathcal{A}\boxtimes_\mathcal{O}\mathcal{B})(U\times V) = \mathcal{A}(U)\otimes_k \mathcal{B}(V)$. This gives $\mathcal{A}\boxtimes_\mathcal{O}\mathcal{B}$ an algebra structure, and it follows that it is a differential $\mathcal{O}_{X\times Y}$-algebra. In particular, it is quasi-coherent with respect to right multiplication, so we would have gotten the same algebra if we had instead defined $\mathcal{A}\boxtimes_\mathcal{O}\mathcal{B}$ with respect to right multiplication at the start.

2.1.6. Let $X$ be a smooth scheme. To properly state Beilinson-Bernstein localization, we must consider twisted differential operators instead of just $\mathcal{D}_X$.

**Definition 2.1.7.** An algebra of twisted differential operators (tdo) is an $\mathcal{O}_X$-differential algebra $\mathcal{D}$ with the following properties:

(i) $\mathcal{O}_X \to F_0\mathcal{D}$ is an isomorphism.
(ii) The morphism $\operatorname{gr}^F_1 \mathcal{D} \to \Theta_X = \mathcal{D}er_k(\mathcal{O}_X,\mathcal{O}_X)$ sending $\xi \mapsto [\xi,\bullet]$ for $\xi \in F_1\mathcal{D}$ is surjective.

In (ii), the commutativity of $\operatorname{gr}^F\mathcal{D}$ implies that $[\xi, f]$ lies in $F_0\mathcal{D} = \mathcal{O}_X$ for $f \in \mathcal{O}_X$.

**Example 2.1.8.** Let $\mathcal{L}$ be a line bundle on $X$. Then $\mathcal{D}_\mathcal{L}$ (see Example 2.1.5(i)) is a tdo. Conditions (i) and (ii) of Definition 2.1.7 are local, so we may assume $\mathcal{L} = \mathcal{O}_X$. In this case, $F_0\mathcal{D}_X = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{O}_X) \simeq \mathcal{O}_X$, and the inclusion $\mathcal{D}er_k(\mathcal{O}_X,\mathcal{O}_X) \subset \mathcal{E}nd_k(\mathcal{O}_X)$ lies in $F_1\mathcal{D}_X$. So we in fact have a decomposition $F_1\mathcal{D}_X = \mathcal{O}_X \oplus \Theta_X$ as left $\mathcal{O}_X$-modules.

**Lemma 2.1.9.** *Let $\mathcal{D}$ be a tdo. Then $\operatorname{gr}^F_1\mathcal{D} \to \Theta_X$ is an isomorphism, and the natural morphism of $\mathcal{O}_X$-algebras $\operatorname{Sym}_{\mathcal{O}_X}(\operatorname{gr}^F_1\mathcal{D}) \to \operatorname{gr}^F\mathcal{D}$ is an isomorphism.*

*Proof.* The injectivity of $\operatorname{gr}_1 \mathcal{D} = F_1\mathcal{D}/F_0\mathcal{D} \to \Theta_X$ follows from the definition of $F_0\mathcal{D}$. We prove that $\operatorname{Sym}^n_{\mathcal{O}_X}(\operatorname{gr}_1 \mathcal{D}) \xrightarrow{\sim} \operatorname{gr}_n \mathcal{D}$ by induction on $n$. The claim is local, so we will assume $X$ is affine and $\Omega^1_X$ is a free $\mathcal{O}_X$-module with basis $dx_1, \ldots, dx_m$ for $x_i \in \Gamma(X, \mathcal{O}_X)$ where $m = \dim X$ (here we use that $X$ is smooth). For $P \in F_n\mathcal{D}$, we have that $[P, \bullet]$ is a $k$-linear morphism $\mathcal{O}_X = F_0\mathcal{D} \to F_{n-1}\mathcal{D}$. If we compose this with the projection $F_{n-1}\mathcal{D} \to \operatorname{gr}_{n-1}\mathcal{D}$, we get a derivation

$$[P, \bullet] \in \mathcal{D}er_k(\mathcal{O}_X, \operatorname{gr}_{n-1}\mathcal{D}) \simeq \Theta_X \underset{\mathcal{O}_X}{\otimes} \operatorname{gr}_{n-1}\mathcal{D}.$$

By induction hypothesis, $\operatorname{gr}_{n-1}\mathcal{D} \xleftarrow{\sim} \operatorname{Sym}^{n-1}(\operatorname{gr}_1 \mathcal{D})$. Combining with $\operatorname{gr}_1 \mathcal{D} \simeq \Theta_X$, we get a morphism

$$\operatorname{gr}_n \mathcal{D} \hookrightarrow \operatorname{gr}_1 \mathcal{D} \underset{\mathcal{O}_X}{\otimes} \operatorname{Sym}^{n-1}(\operatorname{gr}_1 \mathcal{D}) \to \operatorname{Sym}^n(\operatorname{gr}_1 \mathcal{D})$$

where the second map is multiplication. For $\xi_1, \ldots, \xi_n \in F_1\mathcal{D}$ and $f \in \mathcal{O}_X$, we have that $[\xi_1 \cdots \xi_n, f] = \Sigma_{i=1}^n \xi_1 \cdots [\xi_i, f] \cdots \xi_n$. Thus $\xi_1 \cdots \xi_n \mapsto \Sigma_{i=1}^n \xi_i \otimes (\xi_1 \cdots \widehat{\xi_i} \cdots \xi_n) \mapsto n\xi_1 \cdots \xi_n$ under the above morphism. Take $\partial_i \in F_1\mathcal{D}$ such that $[\partial_i, x_j] = \delta_{ij}$. For $P \in F_n\mathcal{D}$, the derivation $[P, \bullet]$ corresponds to

$$\sum_{i=1}^{m} \partial_i \otimes [P, x_i] \in \operatorname{gr}_1 \mathcal{D} \underset{\mathcal{O}_X}{\otimes} \operatorname{gr}_{n-1}\mathcal{D}.$$

We have $[\partial_i[P, x_i], x_j] = \delta_{ij}[P, x_i] + \partial_i[[P, x_i], x_j]$ and $[[P, x_i], x_j] = [[P, x_j], x_i]$. By induction we see that $\Sigma_{i=1}^m \partial_i[[P, x_j], x_i] = (n-1)[P, x_j] \in \operatorname{gr}_{n-1}\mathcal{D}$. The previous calculation implies that $\Sigma_{i=1}^m \partial_i[P, x_i]$ and $nP$ coincide as derivations in $\mathcal{D}er_k(\mathcal{O}_X, \operatorname{gr}_{n-1}\mathcal{D})$. Therefore they must also be equal in $\operatorname{gr}_n \mathcal{D}$. Since we are in characteristic 0, this proves the desired isomorphism. □

*Remark* 2.1.10. Since $\Theta_X$ is locally free, by inductively taking splittings of the grading and applying Lemma 2.1.9, we see that $\mathcal{D}$ is a locally free $\mathcal{O}_X$-module (of infinite rank) with respect to either the left or right action. This observation, together with the proof of Proposition 2.1.3, shows that for a quasi-coherent sheaf $\mathcal{F}$ on a smooth $X$, we have a canonical isomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}if(\mathcal{O}_X, \mathcal{F})$.

2.1.11. Define a morphism of tdo's to be a morphism of $\mathcal{O}_X$-algebras. By passing to the associated graded, we deduce that any morphism of tdo's is an isomorphism. In other words, the tdo's on $X$ form a groupoid.

2.1.12. A *Lie algebroid* $\mathcal{L}$ on $X$ is a quasi-coherent $\mathcal{O}_X$-module equipped with a morphism of $\mathcal{O}_X$-modules $\sigma : \mathcal{L} \to \Theta_X$ and a $k$-linear pairing $[\bullet, \bullet] : \mathcal{L} \otimes_k \mathcal{L} \to \mathcal{L}$ such that

- (i) $[\bullet, \bullet]$ is a Lie algebra bracket and $\sigma$ is a map of Lie algebras,
- (ii) $[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + \sigma(\xi_1)(f)\xi_2$ for $\xi_i \in \mathcal{L}$, $f \in \mathcal{O}_X$.

Lie algebroids form a category in an obvious way. The tangent sheaf $\Theta_X$ is a Lie algebroid (with $\sigma = \mathrm{id}_{\Theta_X}$). For any Lie algebroid $\mathcal{L}$ there is then a unique morphism $\mathcal{L} \to \Theta_X$, which we call the *anchor*.

2.1.13. Define the universal enveloping $\mathcal{O}_X$-differential algebra $U(\mathcal{L})$ to be the sheaf of $k$-algebras generated by $\mathcal{O}_X$ and $\mathcal{L}$ modulo the relations

- (i) $\mathcal{O}_X \xrightarrow{i} U(\mathcal{L})$ is a morphism of $k$-algebras,
- (ii) $\mathcal{L} \xrightarrow{j} U(\mathcal{L})$ is a morphism of Lie algebras,
- (iii) $j(f\xi) = i(f)j(\xi)$ and $[j(\xi), i(f)] = i(\sigma(\xi)(f))$ for $\xi \in \mathcal{L}$, $f \in \mathcal{O}_X$.

For a tdo $\mathcal{D}$, observe that $\operatorname{Lie}\mathcal{D} := F_1\mathcal{D}$ with its left $\mathcal{O}_X$-action is a Lie algebroid, which we call the *Lie algebroid of* $\mathcal{D}$. Lemma 2.1.9 says that $\mathcal{D}$ is uniquely determined by $\operatorname{Lie}\mathcal{D}$. Explicitly, we have the following description.

**Proposition 2.1.14.** *A tdo $\mathcal{D}$ is isomorphic to $U(\operatorname{Lie}\mathcal{D})/U(\operatorname{Lie}\mathcal{D})(i(1)-j(1))$ as an $\mathcal{O}_X$-algebra, where $i,j$ are as above.*

*Proof.* Note that $i(1)-j(1)$ is central in $U(\operatorname{Lie}\mathcal{D})$, so the left ideal is automatically a two-sided ideal. So $\mathcal{A} := U(\operatorname{Lie}\mathcal{D})/U(\operatorname{Lie}\mathcal{D})(i(1)-j(1))$ is a $\mathcal{O}_X$-algebra, which we give the ring filtration induced by the universal enveloping algebra. Then $\operatorname{gr}\mathcal{A}$ is commutative, $\operatorname{gr}_1\mathcal{A} = \operatorname{gr}_1\mathcal{D}$, and $\operatorname{Sym}_{\mathcal{O}_X}(\operatorname{gr}_1\mathcal{D}) \to \operatorname{gr}\mathcal{A}$ is surjective. Evidently we have a map of filtered algebras $\mathcal{A}\to\mathcal{D}$. The composition $\operatorname{Sym}_{\mathcal{O}_X}(\operatorname{gr}_1\mathcal{D}) \to \operatorname{gr}\mathcal{A} \to \operatorname{gr}\mathcal{D}$ is an isomorphism by Lemma 2.1.9, so $\operatorname{gr}\mathcal{A}\to\operatorname{gr}\mathcal{D}$ must in fact be an isomorphism. □

2.1.15. Let $\operatorname{Mod}(\mathcal{D})$ denote the abelian category of sheaves of left $\mathcal{D}$-modules, on which we can apply the well-developed general theory of sheaves of rings. We will primarily be concerned with the abelian subcategory $\operatorname{Mod}_{qc}(\mathcal{D})$ of those modules which are quasi-coherent as $\mathcal{O}_X$-modules. Unless otherwise stated, we will simply call these $\mathcal{D}$-modules.

2.1.16. Put $A = \Gamma(X,\mathcal{D})$. As in the case of $\mathcal{O}$-modules, we have a pair of adjoint functors $\operatorname{Loc}_{\mathcal{D}} : \operatorname{Mod}(A) \rightleftarrows \operatorname{Mod}_{qc}(\mathcal{D}) : \Gamma(X,\bullet)$ where $\operatorname{Loc}_{\mathcal{D}}(M) := \mathcal{D}\otimes_{\underline{A}}\underline{M}$. Here $\underline{A},\underline{M}$ are the constant sheaves. A formal argument [HTT08, Proposition 1.4.4] shows:

**Lemma 2.1.17.** *The following are equivalent:*

(i) $\Gamma(X,\bullet)$ *is exact and* $\Gamma(X,\mathcal{M})=0$ *for* $\mathcal{M}\in\operatorname{Mod}_{qc}(\mathcal{D}_X)$ *implies* $\mathcal{M}=0$.
(ii) *The functors* $\Gamma(X,\bullet)$ *and* $\operatorname{Loc}_{\mathcal{D}}$ *are quasi-inverse.*

We say that $X$ is $\mathcal{D}$-*affine* if the above conditions are satisfied. Of course, any affine scheme is $\mathcal{D}$-affine for any tdo $\mathcal{D}$.

2.2. **Functoriality.** Let $\varphi : Y\to X$ be a morphism of smooth schemes and $\mathcal{D}$ a tdo on $X$. Then $\varphi^*\mathcal{D} = \mathcal{O}_Y\otimes_{\varphi^{-1}\mathcal{O}_X}\varphi^{-1}\mathcal{D}$ is a quasi-coherent $\mathcal{O}_Y$-module with a right $\varphi^{-1}\mathcal{D}$-action. Define

$$\varphi^\sharp\mathcal{D} = \mathcal{D}if_{\varphi^{-1}\mathcal{D}}(\varphi^*\mathcal{D},\varphi^*\mathcal{D})$$

to be the sheaf of $\mathcal{O}_Y$-differential operators that commute with the $\varphi^{-1}\mathcal{D}$-action.

**Proposition 2.2.1.** *The algebra $\varphi^\sharp\mathcal{D}$ is a tdo, and we have an isomorphism of $\mathcal{O}_Y$-modules*

$$\operatorname{Lie}(\varphi^\sharp\mathcal{D}) \xrightarrow{\sim} \Theta_Y \underset{\varphi^*\Theta_X}{\times} \varphi^*(\operatorname{Lie}\mathcal{D})$$

*where $d\varphi : \Theta_Y\to\varphi^*\Theta_X$ is the dual of the canonical map $\varphi^*\Omega^1_X\to\Omega^1_Y$.*

*Proof.* First, let us show that $\varphi^\sharp\mathcal{D}$ is quasi-coherent. Observe that the universal property of the tensor product gives an isomorphism of $\mathcal{O}_Y$-bimodules

$$\mathcal{D}if_{\varphi^{-1}\mathcal{D}}(\varphi^*\mathcal{D},\varphi^*\mathcal{D}) \xrightarrow{\sim} \mathcal{D}if_{\varphi^{-1}\mathcal{O}_X}(\mathcal{O}_Y,\varphi^*\mathcal{D}).$$

By Proposition 2.1.3, $\mathcal{M} = \mathcal{D}if_k(\mathcal{O}_Y,\varphi^*\mathcal{D})$ is quasi-coherent with respect to either $\mathcal{O}_Y$-action. The $\mathcal{O}_Y$-actions commute with the $\varphi^{-1}\mathcal{O}_X$-actions on $\mathcal{O}_Y$ and $\varphi^*\mathcal{D}$, so $\mathcal{M}$ has the structure of a $\varphi^{-1}\mathcal{O}_X$-bimodule. Let us suppose that $X=\operatorname{Spec}A$ and $Y=\operatorname{Spec}B$ are affine. Put $M=\Gamma(Y,\mathcal{M})$ and $I_A\subset A\otimes A$ the ideal of the diagonal. Then $\operatorname{Dif}_A(B,\Gamma(Y,\varphi^*\mathcal{D}))\simeq \operatorname{Hom}_{A\otimes A}(A\otimes A/I_A, M)$. Since $A\otimes A/I_A$ is a finitely presented $(A\otimes A)$-module, localization over $B\otimes B$ commutes with the Hom. Quasi-coherence of $\mathcal{M}$ now implies that $\varphi^\sharp\mathcal{D}$ is quasi-coherent and thus an $\mathcal{O}_Y$-differential algebra.

We next prove that $\mathcal{O}_Y \xrightarrow{\sim} F_0(\varphi^\sharp\mathcal{D})$. Working locally, assume that $\Omega^1_X$ has basis $dx_1,\dots,dx_n$ for $x_i\in\Gamma(X,\mathcal{O}_X)$ and let $\xi_i\in F_1\mathcal{D}$ map to the dual basis in $\Theta_X$. Take $\widetilde{P}\in F_0(\varphi^\sharp\mathcal{D}) = \mathcal{H}om_{\mathcal{O}_Y,\varphi^{-1}\mathcal{D}}(\varphi^*\mathcal{D},\varphi^*\mathcal{D})$. Then $\widetilde{P}(1\otimes 1)\in\varphi^*\mathcal{D}$ commutes with $\varphi^{-1}\mathcal{O}_X$. Lemma 2.1.9 implies that $\widetilde{P}(1\otimes 1) = P(\xi_1,\dots,\xi_n)$ for some polynomial $P$ on $n$ indeterminants with coefficients in $\mathcal{O}_Y$. On polynomials, one sees that $-\operatorname{ad} x_i(P(\xi_1,\dots,\xi_n))$ is simply the partial derivative $\partial/\partial\xi_i$. Since $P(\xi_1,\dots,\xi_n)$ commutes with all $x_i$, the polynomial $P$ must be constant. Hence $\widetilde{P}(1\otimes 1)\in\mathcal{O}_Y\subset\varphi^*\mathcal{D}$.

The morphism $F_1(\varphi^\sharp\mathcal{D})\to\Theta_Y$ is now well-defined. Take $\widetilde{P}\in F_1(\varphi^\sharp\mathcal{D})$. For $b\in\mathcal{O}_Y$, we have

$$\widetilde{P}(b\otimes 1)=b\widetilde{P}(1\otimes 1)+[\widetilde{P},b](1\otimes 1). \tag{2.2.1.1}$$

If $a\in\varphi^{-1}\mathcal{O}_X$, then $\widetilde{P}(\varphi^*(a)\otimes 1)=\widetilde{P}(1\otimes 1)a$ so that $\operatorname{ad} a(\widetilde{P}(1\otimes 1))=-[\widetilde{P},\varphi^*(a)](1\otimes 1)\in\mathcal{O}_Y$. Letting $a=x_i$ for $i=1,\dots,n$ and using the same polynomial argument as in the previous paragraph, we deduce that $\widetilde{P}(1\otimes 1)\in\varphi^*(F_1\mathcal{D})$. The corresponding image $[\widetilde{P}(1\otimes 1),\bullet]$ in $\mathcal{D}er_k(\varphi^{-1}\mathcal{O}_X,\mathcal{O}_Y)\simeq\varphi^*\Theta_X$ coincides with $[\widetilde{P},\varphi^*(\bullet)](1\otimes 1)$. This defines the desired morphism of $\mathcal{O}_Y$-modules $F_1(\varphi^\sharp\mathcal{D})\to\varphi^*(F_1\mathcal{D})\times_{\varphi^*\Theta_X}\Theta_Y$. Equation (2.2.1.1) shows that this morphism is an isomorphism. Since $\varphi^*(F_1\mathcal{D})\to\varphi^*\Theta_X$ is surjective, the projection $\varphi^*(F_1\mathcal{D})\times_{\varphi^*\Theta_X}\Theta_Y\to\Theta_Y$ is also surjective. This proves condition (ii) of Definition 2.1.7, and we conclude that $\varphi^\sharp\mathcal{D}$ is a tdo. $\square$

2.2.2. We call $\varphi^\sharp\mathcal{D}$ the *pullback* of $\mathcal{D}$ along $\varphi$. A good way to think about $\varphi^\sharp\mathcal{D}$ is through the following universal property: for an $\mathcal{O}_Y$-differential algebra $\mathcal{A}$, mapping to $\varphi^\sharp\mathcal{D}$ is the same as providing an $\mathcal{O}_Y$-algebra action $\mathcal{A}\to\mathcal{E}nd_{\varphi^{-1}\mathcal{D}}(\varphi^*\mathcal{D})$ on $\varphi^*\mathcal{D}$ that commutes with the right $\varphi^{-1}\mathcal{D}$ action. Functoriality of $\mathcal{H}om_{\varphi^{-1}\mathcal{O}_X}(\mathcal{O}_Y,\varphi^*(\bullet))$ specifies what $\varphi^\sharp$ does to morphisms. This makes $\varphi^\sharp$ a functor between the groupoids of tdo's on $X$ and $Y$.

2.2.3. Let $\mathcal{L}$ be a Lie algebroid on $X$. Define $\varphi^\sharp\mathcal{L}=\Theta_Y\times_{\varphi^*\Theta_X}\varphi^*\mathcal{L}$ as an $\mathcal{O}_Y$-module. We make $\varphi^\sharp\mathcal{L}$ into a Lie algebroid by letting the anchor be the first projection to $\Theta_Y$ and defining the bracket as

$$\big[(\xi,\textstyle\sum_i f_i\otimes P_i),\ (\eta,\sum_j g_j\otimes Q_j)\big]=\big([\xi,\eta],\ \sum_{i,j} f_ig_j\otimes[P_i,Q_j]+\xi(g_j)\otimes P_i-\eta(f_i)\otimes Q_j\big)$$

for $\xi,\eta\in\Theta_X$, $f_i,g_j\in\mathcal{O}_Y$, and $P_i,Q_j\in\varphi^{-1}\mathcal{L}$. Clearly $\varphi^\sharp$ is a functor. This is the *Lie algebroid pullback*. One can then check using (2.2.1.1) that Proposition 2.2.1 gives an isomorphism $\operatorname{Lie}(\varphi^\sharp\mathcal{D})\simeq\varphi^\sharp(\operatorname{Lie}\mathcal{D})$ of Lie algebroids on $Y$.

**Example 2.2.4.** Let $\mathcal{L}$ be a line bundle on $X$. Then $\varphi^\sharp\mathcal{D}_{\mathcal{L}}\simeq\mathcal{D}_{\varphi^*\mathcal{L}}$. There is a natural map $\operatorname{Lie}(\mathcal{D}_{\varphi^*\mathcal{L}})\to\varphi^*(\operatorname{Lie}\mathcal{D}_{\mathcal{L}})\times_{\varphi^*\Theta_X}\Theta_Y$ induced from restriction. To see that it is an isomorphism, we localize to assume $\mathcal{L}=\mathcal{O}_X$. We observed in Example 2.1.8 that $\operatorname{Lie}\mathcal{D}_X=\mathcal{O}_X\oplus\Theta_X$. Therefore

$$\operatorname{Lie}\mathcal{D}_Y\simeq\mathcal{O}_Y\oplus\Theta_Y\simeq\varphi^*(\mathcal{O}_X\oplus\Theta_X)\underset{\varphi^*\Theta_X}{\times}\Theta_Y.$$

Proposition 2.2.1 then implies that $\operatorname{Lie}(\varphi^\sharp\mathcal{D}_{\mathcal{L}})\simeq\operatorname{Lie}(\mathcal{D}_{\varphi^*\mathcal{L}})$, and the isomorphism of tdo's follows from the explicit description of Proposition 2.1.14.

**Lemma 2.2.5.** *If $\varphi$ is étale, then evaluation at $1\otimes 1$ defines an isomorphism $\varphi^\sharp\mathcal{D}\to\varphi^*\mathcal{D}$.*

*Proof.* This holds in fact for any differential algebra (cf. [BB93, 1.4.5]). We will use the notations and ideas of 2.1.1. Suppose for the moment that $X=\operatorname{Spec}A$ and $Y=\operatorname{Spec}B$ are affine, and let $I_A,I_B$ denote the corresponding kernels of multiplication. Then $\varphi^*\Omega_X\simeq\Omega_Y$ implies that the ring morphism $B\otimes_A(A\otimes A/I_A^n)\xrightarrow{\sim}B\otimes B/I_B^n$ is an isomorphism of $B$-modules. In terms of schemes, this says that $X_\Delta^{(n)}\times_{X\times X}(Y\times X)\simeq Y_\Delta^{(n)}\simeq X_\Delta^{(n)}\times_{X\times X}(X\times Y)$. We deduce that $\varphi^*\mathcal{D}\simeq\mathcal{D}\otimes_{\varphi^{-1}\mathcal{O}_X}\mathcal{O}_Y$ has the structure of a differential $\mathcal{O}_Y$-module, which induces an algebra structure. Left multiplication consequently defines a morphism of $\mathcal{O}_Y$-algebras $\varphi^*\mathcal{D}\to\varphi^\sharp\mathcal{D}$.

Let us return to the affine situation and put $R = \Gamma(X, \mathcal{D})$. Since $A \otimes A/I_A^n$ is finitely presented as a left $A$-module, we have canonical isomorphisms

$$B \underset{A}{\otimes} \mathrm{Hom}_{A \otimes A}(A \otimes A/I_A^n, R) \xrightarrow{\sim} \mathrm{Hom}_{B \otimes A}(B \underset{A}{\otimes}(A \otimes A/I_A^n), B \underset{A}{\otimes} R)$$
$$\xleftarrow{\sim} \mathrm{Hom}_{B \otimes A}(B \otimes B/I_B^n, B \underset{A}{\otimes} R) = \mathrm{Dif}_A^n(B, B \underset{A}{\otimes} R).$$

This coincides with the morphism $\varphi^*(F_n\mathcal{D}) \to F_n(\varphi^\sharp\mathcal{D})$, so we are done. Consequently, evaluation at $1 \otimes 1$ from $\varphi^\sharp\mathcal{D} \to \varphi^*\mathcal{D}$ is a ring morphism. □

**Example 2.2.6.** Suppose $\varphi : Y = T \times X \to X$ is projection to the second factor. Then $\varphi^*\mathcal{D} = \mathcal{O}_T \boxtimes_{\mathcal{O}} \mathcal{D}$, and we have an isomorphism $\mathcal{D}_T \boxtimes_{\mathcal{O}} \mathcal{D} \xrightarrow{\sim} \varphi^\sharp\mathcal{D}$ sending $\partial \otimes a \in \mathcal{D}_T \boxtimes \mathcal{D}$ to the operator $t \otimes b \mapsto \partial(t) \otimes ab$ (see Example 2.1.5).

2.2.7. Our definition of $\varphi^\sharp$ gives $\varphi^*\mathcal{D}$ the structure of a $(\varphi^\sharp\mathcal{D}, \varphi^{-1}\mathcal{D})$-bimodule. If $\mathcal{M}$ is a $\mathcal{D}$-module, then the pullback

$$\varphi^*\mathcal{M} = \mathcal{O}_Y \underset{\varphi^{-1}\mathcal{O}_X}{\otimes} \varphi^{-1}\mathcal{M} \simeq \varphi^*\mathcal{D} \underset{\varphi^{-1}\mathcal{D}}{\otimes} \varphi^{-1}\mathcal{M}$$

has the structure of a $\varphi^\sharp\mathcal{D}$-module.

2.2.8. Let $\psi : Z \to Y$ be another morphism of smooth schemes. Then the above discussion gives $\psi^*\varphi^*\mathcal{D}$ the structure of a $\psi^\sharp\varphi^\sharp\mathcal{D}$-module. Since $\psi^*\varphi^*\mathcal{D} \simeq (\varphi \circ \psi)^*\mathcal{D}$, we get a canonical morphism of $\mathcal{O}_Z$-algebras $\psi^\sharp\varphi^\sharp\mathcal{D} \to \mathcal{E}nd_k((\varphi \circ \psi)^*\mathcal{D})$, which must land in the differential part. The $\psi^\sharp\varphi^\sharp\mathcal{D}$-module structure comes from tensoring on the left, so it is compatible with the right $(\varphi \circ \psi)^{-1}\mathcal{D}$-action. Therefore we get a morphism of tdo's $c_{\varphi,\psi} : \psi^\sharp\varphi^\sharp\mathcal{D} \to (\varphi \circ \psi)^\sharp\mathcal{D}$, which is necessarily an isomorphism (see Remark 2.1.11).

The $c_{\varphi,\psi}$ satisfy the relevant associativity axioms. In other words, pullback makes tdo's a category fibered in groupoids over the small site $X_{\mathrm{sm}}$ of schemes smooth over $X$.

Note that $c_{\varphi,\psi}$ is defined so that the canonical isomorphism $\psi^*\varphi^*\mathcal{D} \simeq (\varphi\circ\psi)^*\mathcal{D}$ is a morphism of $\psi^\sharp\varphi^\sharp\mathcal{D}$-modules, where the action on the RHS comes from $c_{\varphi,\psi}$.

2.3. **Smooth descent.** We now have all the ingredients necessarily to show that tdo's satisfy the property of smooth descent. To be precise:

**Lemma 2.3.1.** *Pullback defines an equivalence of categories between tdo's on $X$ and the category of descent datum $(Y, \mathcal{B}, \tau)$ where $Y \in X_{\mathrm{sm}}$, $\mathcal{B}$ is a tdo on $Y$, and $\tau : \mathrm{pr}_1^\sharp\mathcal{B} \xrightarrow{\sim} \mathrm{pr}_2^\sharp\mathcal{B}$ is an isomorphism satisfying the cocycle condition $\mathrm{pr}_{13}^\sharp(\tau) = \mathrm{pr}_{23}^\sharp(\tau) \circ \mathrm{pr}_{12}^\sharp(\tau)$.*

*Proof.* Cf. [BB93, Lemma 1.5.4]. Let $(Y \xrightarrow{\varphi} X, \mathcal{B}, \tau)$ be a descent datum. We would like to specify a tdo $\mathcal{A}$ on $X$ such that $\varphi^\sharp\mathcal{A} \simeq \mathcal{B}$ in a natural way. First suppose that $\varphi$ admits a section $s : X \to Y$ such that $\varphi \circ s = \mathrm{id}_X$. Then we can define $\mathcal{A} = s^\sharp\mathcal{B}$. Under the correspondence $X \times_{Y,\mathrm{pr}_1} (Y \times_X Y) \simeq Y$, the transition morphism $\tau$ pulls back to an isomorphism

$$\varphi^\sharp\mathcal{A} \simeq (s \circ \varphi)^\sharp\mathcal{B} \xrightarrow{\sim} \mathrm{id}_Y^\sharp \mathcal{B}.$$

The cocycle condition ensures that everything is canonical.

Any smooth morphism admits a section étale locally, so we have reduced to the case when $\varphi$ is étale. The proof of Lemma 2.2.5 shows that $\tau$ in fact gives us descent datum of quasi-coherent sheaves $(Y_\Delta^{(n)} \to X_\Delta^{(n)}, F_n\mathcal{B}, \tau)$. By faithfully flat descent of quasi-coherent sheaves [FGI+05, Theorem 4.23], we get a differential $\mathcal{O}_X$-module $\mathcal{A}$ with $\varphi^*\mathcal{A} \simeq \mathcal{B}$. The algebra structure on $\mathcal{A}$ is recovered by descent from $\mathcal{B} \otimes_{\mathcal{O}_Y} \mathcal{B} \simeq \varphi^*(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}) \to \varphi^*\mathcal{A} \simeq \mathcal{B}$. This makes $\mathcal{A}$ a tdo, and it is immediate that $(Y, \mathcal{B}, \tau) \mapsto \mathcal{A}$ defines a functor quasi-inverse to pullback. □

In other words, tdo's form a stack on the small site $X_{\mathrm{sm}}$.

2.3.2. The argument used to prove the lemma also shows that the categories $\mathrm{Mod}_{qc}(\varphi^\sharp\mathcal{D})$, for smooth morphisms $Y \xrightarrow{\varphi} X$, form a stack over $X_{\mathrm{sm}}$. In particular, for $\mathcal{D} = \mathcal{D}_X$, we get that $\mathrm{Mod}_{qc}(\mathcal{D}_Y)$ for $Y \in X_{\mathrm{sm}}$ defines a stack.

2.3.3. **Differential operators on algebraic stacks.** The smooth descent property allows us to define tdo's on algebraic stacks, which will provide a nice framework for our later discussions[1].

Let $\mathcal{X}$ be a smooth algebraic $k$-stack with schematic diagonal. A *tdo on* $\mathcal{X}$ consists of the following data (cf. [BB93, 1.7]):

(i) For any scheme $X$ and a smooth 1-morphism $\pi : X \to \mathcal{X}$ one has a tdo $\mathcal{D}_{(X,\pi)}$ on $X$.

(ii) For any $(X,\pi)$, $(X',\pi')$ as above, a morphism $\alpha : X' \to X$, and a 2-morphism $\pi' \xrightarrow{\widetilde{\alpha}} \pi\alpha$, one has an isomorphism $\widetilde{\alpha}_{\mathcal{D}} : \mathcal{D}_{(X',\pi')} \xrightarrow{\sim} \alpha^\sharp\mathcal{D}_{(X,\pi)}$.

We require the $\widetilde{\alpha}_{\mathcal{D}}$ to satisfy the natural compatibilities with compositions of $(\alpha,\widetilde{\alpha})$'s.

Choose a smooth surjective 1-morphism $\pi : X \to \mathcal{X}$. Then the smooth descent property of tdo's implies that to define a tdo on $\mathcal{X}$, it suffices to specify *a single tdo* $\mathcal{D} = \mathcal{D}_{(X,\pi)}$ on $X$ and an isomorphism $\gamma_{\mathcal{D}} : \mathrm{pr}_1^\sharp\mathcal{D} \xrightarrow{\sim} \mathrm{pr}_2^\sharp\mathcal{D}$ satisfying the cocycle condition (observe that $X \times_{\mathcal{X}} X$ is representable by a scheme since $\mathcal{X}$ has schematic diagonal). This is the crucial point that we will make use of later.

Given a tdo $\mathcal{D}$ on $\mathcal{X}$, we define a $\mathcal{D}$-module on $\mathcal{X}$ to be a collection of $\mathcal{D}_{(X,\pi)}$-modules $\mathcal{M}_{(X,\pi)}$ together with $\widetilde{\alpha}_{\mathcal{D}}$-isomorphisms $\mathcal{M}_{(X',\pi')} \xrightarrow{\sim} \alpha^*\mathcal{M}_{(X,\pi)}$ compatible with compositions. Again, smooth descent says that this is equivalent to giving, for a single smooth covering $X \xrightarrow{\pi} \mathcal{X}$, a $\mathcal{D}$-module $\mathcal{M}$ on $X$ together with a $\gamma_{\mathcal{D}}$-isomorphism $\tau_{\mathcal{M}} : \mathrm{pr}_1^*\mathcal{M} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{M}$ that satisfies the cocycle condition. By a $\gamma_{\mathcal{D}}$-morphism we mean that $\tau_{\mathcal{M}}$ is a morphism of $\mathrm{pr}_1^\sharp\mathcal{D}$-modules, where $\mathrm{pr}_1^\sharp\mathcal{D}$ acts on $\mathrm{pr}_2^*\mathcal{M}$ via $\gamma_{\mathcal{D}}$. The $\mathcal{D}$-modules on $\mathcal{X}$ form an abelian category $\mathrm{Mod}_{qc}(\mathcal{X},\mathcal{D})$.

## 3. Operations on $\mathcal{D}$-modules

*All schemes in this section will be assumed to be smooth, quasi-projective*[2]*, and of pure dimension.* In the next two sections, we give an overview of the basic definitions and results concerning $\mathcal{D}$-modules when $\mathcal{D} = \mathcal{D}_X$ is the sheaf of (ordinary) differential operators on a scheme $X$. The organization and content heavily follow [Ber84]. We also consulted [BGK$^+$87], [HTT08], and [Kas03] to help fill in details of proofs.

3.0.1. Let us first summarize some properties of $\mathcal{D}_X$ from the discussion of §2. We have the filtration $F_i\mathcal{D}_X = \mathcal{D}if^i(\mathcal{O}_X,\mathcal{O}_X)$ on $\mathcal{D}_X$, called the *order filtration*. The sheaf of $\mathcal{O}_X$-algebras $\mathcal{D}_X$ is quasi-coherent with respect to either left or right multiplication. There is a decomposition $F_1\mathcal{D}_X = \mathcal{O}_X \oplus \Theta_X$ of left $\mathcal{O}_X$-modules, and $\Theta_X \subset \mathcal{D}_X$ induces an isomorphism of commutative $\mathcal{O}_X$-algebras $\mathrm{Sym}_{\mathcal{O}_X}(\Theta_X) \simeq \mathrm{gr}^F\mathcal{D}_X$ by Lemma 2.1.9. Proposition 2.1.14 gives an explicit description of $\mathcal{D}_X$ as the tensor algebra of $\Theta_X$ over $\mathcal{O}_X$ modulo the relations $[\xi, f] = \xi(f)$ for $\xi \in \Theta_X$ and $f \in \mathcal{O}_X$.

Since $X$ is smooth of pure dimension $n := \dim X$, for any $x \in X$ there exists an affine neighborhood $U$ of $x$ such that $\Omega_X^1|_U = \Omega_U^1$ is free with basis $dx_1,\ldots,dx_n$ for $x_i \in \Gamma(U,\mathcal{O}_X)$.

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[1]The use of stacks introduces no new content; we just find that it gives a useful perspective.

[2]Quasi-projectivity is not necessary for anything not involving derived categories; finite type is enough.

Let $\partial_1, \ldots, \partial_n$ be the dual basis of vector fields in $\Theta_U$, i.e., $\partial_i(x_j) = \delta_{ij}$. We call $(x_i, \partial_i)$ a *coordinate system* of $U$. The vector fields commute in $\mathcal{D}_U$, and we have

$$\mathcal{D}_U \simeq \bigoplus_{\alpha \in \mathbf{N}_0^n} \mathcal{O}_U \partial^\alpha \quad (\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n})$$

as a left $\mathcal{O}_X$-module. Any element $P \in \mathcal{D}_U$ can then be uniquely written in the form $\Sigma f_\alpha \partial^\alpha$ for $f_\alpha \in \mathcal{O}_U$. We can make the analogous statements for $\mathcal{D}_U$ as a right $\mathcal{O}_X$-module.

3.1. **Left and right $\mathcal{D}$-modules.** We will work in the abelian category $\mathrm{Mod}_{qc}(\mathcal{D}_X)$ of sheaves of left $\mathcal{D}_X$-modules that are quasi-coherent as $\mathcal{O}_X$-modules, and we will simply call these $\mathcal{D}_X$-modules unless otherwise specified. We analogously define the category $\mathrm{Mod}_{qc}(\mathcal{D}_X^{\mathrm{op}})$ of right $\mathcal{D}_X$-modules.

**Example 3.1.1.** We may equip $\mathcal{D}_X$ itself with either a left or right $\mathcal{D}_X$-module structure via multiplication. By definition of differential operators, the structure sheaf $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module.

3.1.2. Proposition 2.1.14 gives us a more explicit way of defining $\mathcal{D}_X$-modules: Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Then giving a left (resp. right) $\mathcal{D}_X$-module structure on $\mathcal{M}$ extending the $\mathcal{O}_X$-module structure is equivalent to giving a $k$-linear morphism $\Theta_X \to \mathcal{E}nd_k(\mathcal{M}) : \xi \mapsto \nabla_\xi$ satisfying the conditions:

(i) $\nabla_{f\xi}(m) = f\nabla_\xi(m)$ (resp. $\nabla_{f\xi}(m) = \nabla_\xi(fm)$),
(ii) $\nabla_\xi(fm) = \xi(f)m + f\nabla_\xi(m)$,
(iii) $\nabla_{[\xi_1,\xi_2]}(m) = [\nabla_{\xi_1}, \nabla_{\xi_2}](m)$

for all $f \in \mathcal{O}_X$, $\xi \in \Theta_X$, and $m \in \mathcal{M}$. Given $\nabla$, the $\mathcal{D}_X$-action on $\mathcal{M}$ is defined by $\xi m = \nabla_\xi(m)$ (resp. $m\xi = -\nabla_\xi(m)$. Note the sign difference).

3.1.3. Let $\Omega_X^n = \wedge^n \Omega_X^1$ denote the canonical sheaf on $X$ where $n = \dim X$. By considering the isomorphism $\Omega_X^n \simeq \mathcal{H}om_{\mathcal{O}_X}(\wedge^n \Theta_X, \mathcal{O}_X)$, we have a natural action of $\Theta_X$ on $\Omega_X^n$ called the *Lie derivative*, which is defined by

$$(\mathrm{Lie}_\xi\, \omega)(\xi_1, \ldots, \xi_n) = \xi(\omega(\xi_1, \ldots, \xi_n)) - \sum_{i=1}^n \omega(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_n)$$

for $\omega \in \Omega_X^n$ and $\xi, \xi_1, \ldots, \xi_n \in \Theta_X$. One can now check the conditions of 3.1.2 to see that $\omega\xi := -\mathrm{Lie}_\xi\, \omega$ defines a *right* $\mathcal{D}_X$-module structure on $\Omega_X^n$. Condition (iii) relies on the fact that $\Omega_X^n$ consists of differential forms of top degree.

We may also define some basic tensor and Hom operations over $\mathcal{O}_X$ using 3.1.2.

**Proposition 3.1.4.** *Let $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$ and $\mathcal{M}', \mathcal{N}' \in \mathrm{Mod}_{qc}(\mathcal{D}_X^{\mathrm{op}})$. Then we have*

| | |
|---|---|
| $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$ | $\xi(m \otimes \ell) = (\xi m) \otimes \ell + m \otimes \xi\ell,$ |
| $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{D}_X^{\mathrm{op}})$ | $(m' \otimes \ell)\xi = m'\xi \otimes \ell - m' \otimes \xi\ell,$ |
| $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$ | $(\xi\phi)(m) = \xi(\phi(m)) - \phi(\xi m),$ |
| $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}') \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$ | $(\xi\phi)(m) = -\phi(m)\xi + \phi(m\xi),$ |
| $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}') \in \mathrm{Mod}_{qc}(\mathcal{D}_X^{\mathrm{op}})$ | $(\phi\xi)(m) = \phi(m)\xi + \phi(\xi m)$ |

*where $\xi \in \Theta_X, m \in \mathcal{M}, \ell \in \mathcal{N}, m' \in \mathcal{M}'$, and $\phi$ is a morphism between the relevant modules. This makes $\mathrm{Mod}_{qc}(\mathcal{D}_X)$ into a tensor category.*

*Remark* 3.1.5. An easy way to remember the consequences of Proposition 3.1.4 is *Oda's rule* (cf. [HTT08, Remark 1.2.10]), which says that a line bundle on a smooth curve of genus $g$ is a left (resp. right) $\mathcal{D}_X$-module if and only if it is of degree 0 (resp. $2g-2$). By considering "left"$=0$, "right"$=1$ and $\otimes \leftrightarrow +$, $\mathcal{H}om(\bullet, \bigstar) = -\bullet + \bigstar$, we get the right answer in determining if the action is on the left or right. Oda's rule also shows that there is no $\mathcal{D}_X$-module structure on $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}'$.

3.1.6. By considering $\mathcal{D}_X$ as a left $\mathcal{D}_X$-module, we get a right $\mathcal{D}_X$-module

$$^\Omega\mathcal{D}_X := \Omega_X^n \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X.$$

It has two commuting right $\mathcal{D}_X$-actions: one from the tensor product and another from right multiplications on $\mathcal{D}_X$. In other words, $^\Omega\mathcal{D}_X$ is a $(\mathcal{D}_X^{\mathrm{op}}, \mathcal{D}_X)$-bimodule.

There is a unique $k$-linear involution $v$ of $^\Omega\mathcal{D}_X$ that interchanges the two right $\mathcal{D}_X$-module structures and is the identity on $\Omega_X^n \subset {}^\Omega\mathcal{D}_X$. Explicitly $v$ is defined by

$$v(\omega \otimes \xi_1 \cdots \xi_i) = \Sigma(-1)^m \omega\xi_{j_1} \cdots \xi_{j_{n-m}} \otimes \xi_{\ell_m} \cdots \xi_{\ell_1}$$

for $\omega \in \Omega_X^n$ and $\xi_1, \ldots, \xi_i \in \Theta_X$, where the sum is over all $\{j_1 < \cdots < j_{n-m}\} \sqcup \{\ell_1 < \cdots < \ell_m\} = \{1, \ldots, n\}$ and we appeal to the description of $\mathcal{D}_X$ mentioned in 3.0.1. A computation shows that $v$ is an involution, and the definition was forced such that it interchanges the two right $\mathcal{D}_X$-module structures. Warning: the two $\mathcal{O}_X$-actions on $^\Omega\mathcal{D}_X$ are not the same; they correspond to the $\mathcal{O}_X$-bimodule structure on $\mathcal{D}_X$.

Define $\mathcal{D}_X^\Omega = \mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}({}^\Omega\mathcal{D}_X, \mathcal{D}_X)$ where we consider $^\Omega\mathcal{D}_X$ with its first structure. Then $\mathcal{D}_X^\Omega$ has two commuting left $\mathcal{D}_X$-module structures: one from left multiplications on $\mathcal{D}_X$ and another from the second right $\mathcal{D}_X$-module structure on $^\Omega\mathcal{D}_X$. We will suggestively say that $\mathcal{D}_X^\Omega$ is a $(\mathcal{D}_X, \mathcal{D}_X^{\mathrm{op}})$-bimodule.

Thinking of $\mathcal{D}_X^{\mathrm{op}}$ and $\mathcal{D}_X$ as distinct algebras, the $(\mathcal{D}_X^{\mathrm{op}}, \mathcal{D}_X)$-bimodule structure on $^\Omega\mathcal{D}_X$ gives us the usual $\otimes$, $\mathcal{H}om$ adjunction (of bimodules over noncommutative rings)

$$\mathrm{Hom}_{\mathcal{D}_X^{\mathrm{op}}}\left({}^\Omega\mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{M}, \mathcal{M}'\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_X}\left(\mathcal{M}, \mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}({}^\Omega\mathcal{D}_X, \mathcal{M}')\right)$$

for $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$ and $\mathcal{M}' \in \mathrm{Mod}_{qc}(\mathcal{D}_X^{\mathrm{op}})$. In other words, we have a pair of adjoint functors $\overrightarrow{\Omega} : \mathrm{Mod}_{qc}(\mathcal{D}_X) \rightleftarrows \mathrm{Mod}_{qc}(\mathcal{D}_X^{\mathrm{op}}) : \overleftarrow{\Omega}$ defined by

$$\overrightarrow{\Omega}(\mathcal{M}) = {}^\Omega\mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{M} \simeq \Omega_X^n \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \quad \text{and} \quad \overleftarrow{\Omega}(\mathcal{M}') = \mathcal{H}om_{\mathcal{D}_X^{\mathrm{op}}}({}^\Omega\mathcal{D}_X, \mathcal{M}').$$

**Proposition 3.1.7.** *The functors $\overrightarrow{\Omega}$ and $\overleftarrow{\Omega}$ are quasi-inverse. Furthermore, the natural map*

$$\mathcal{M}' \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_X^\Omega \xrightarrow{\sim} \overleftarrow{\Omega}(\mathcal{M}')$$

*is an isomorphism.*

*Proof.* We must check that certain natural morphisms are isomorphisms in the categories of left and right $\mathcal{D}_X$-modules. By applying the involution $v$, which is the identity on $\Omega_X^n \subset {}^\Omega\Omega_X$, we reduce to showing certain isomorphisms in the category of $\mathcal{O}_X$-modules. There everything follows from $\Omega_X^n$ being a line bundle. □

Warning: Oda's rule tells us that $\Omega_X^{-n}$ does not have a $\mathcal{D}_X$-module structure, so we will try to use $\mathcal{D}_X^\Omega$ instead of $\Omega_X^{-n}$ whenever possible when working with $\mathcal{D}_X$-modules.

*Remark* 3.1.8. Since $\Omega_X^n$ is locally free of rank 1 as an $\mathcal{O}_X$-module, we see that $^\Omega\mathcal{D}_X$ with its second structure is locally free of rank 1 as a right $\mathcal{D}_X$-module. Applying the involution $v$, we deduce that $^\Omega\mathcal{D}_X$ is also locally free if considered with its first structure. Similarly, $\mathcal{D}_X^\Omega$ is locally free of rank 1 as a left $\mathcal{D}_X$-module. In particular, we deduce that $\overrightarrow{\Omega}$ and $\overleftarrow{\Omega}$ are exact functors.

**Example 3.1.9.** Suppose that $X$ is affine with a coordinate system $(x_i, \partial_i)$. We have a trivialization $\mathcal{O}_X \xrightarrow{\sim} \Omega_X^n$ via the top form $\nu = dx_1 \cdots dx_n$. For $P = \Sigma f_\alpha \partial^\alpha \in \mathcal{D}_X$, define the *formal adjoint* ${}^tP = \Sigma(-\partial)^\alpha f_\alpha$, where $(-\partial)^\alpha = (-1)^{\alpha_1 + \cdots + \alpha_n}\partial^\alpha$. Since $\mathrm{Lie}_{\partial_i}(\nu) = 0$ and the $\partial_i$ commute, we see that $\overrightarrow{\Omega}(\mathcal{M})$ takes the same underlying $\mathcal{O}_X$-module and gives it a right $\mathcal{D}_X$-action by $mP := ({}^tP)m$. This in particular shows us what the bimodule structure on ${}^\Omega\mathcal{D}_X$ looks like. The involution $v$ now coincides with the ring anti-isomorphism $P \mapsto {}^tP$ of $\mathcal{D}_X$. It follows that for $Q = \Sigma\, \partial^\alpha f_\alpha \in \mathcal{D}_X$, the formal adjoint ${}^tQ = \Sigma\, f_\alpha(-\partial)^\alpha$. On the other side, we see that $\overleftarrow{\Omega}(\mathcal{M}')$ takes the same underlying $\mathcal{O}_X$-module and gives it a left $\mathcal{D}_X$-action by $Pm' := m'({}^tP)$. This gives the bimodule structure on $\mathcal{D}_X^\Omega$.

3.2. **Derived categories.** As we will see shortly, in order for some operations on $\mathcal{D}$-modules to make sense, we must make full use of derived categories (cf. [GM03]). In this section, we use some (noncommutative) algebra and algebraic geometry to show that the triangulated category $D^b(\mathrm{Mod}_{qc}(\mathcal{D}_X))$ of complexes with bounded cohomology is "nice" enough for us to work in.

For an arbitrary sheaf of rings $\mathcal{A}$ on $X$, let $D(\mathcal{A})$ denote the derived category $D(\mathrm{Mod}(\mathcal{A}))$. Good references on the general theory of $D(\mathcal{A})$ are [KS94] and [HTT08, Appendix C]). In some of the intermediate steps of our arguments, we will encounter non-quasi-coherent sheaves, so one might wonder if we should work in $D(\mathcal{D}_X)$ or $D(\mathrm{Mod}_{qc}(\mathcal{D}_X))$. Luckily, we have the following theorem [BGK+87, VI, Theorem 2.10] of Bernstein:

**Theorem 3.2.1** (J. Bernstein)**.** *Let $X$ be a noetherian separated scheme. Suppose we have a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ on $X$ that is quasi-coherent as a left $\mathcal{O}_X$-module. Let $D^b_{qc}(\mathrm{Mod}(\mathcal{A})) \subset D(\mathcal{A})$ denote the full triangulated subcategory of complexes with bounded quasi-coherent cohomology. Then the natural morphism of derived categories*

$$D^b(\mathrm{Mod}_{qc}(\mathcal{A})) \to D^b_{qc}(\mathrm{Mod}(\mathcal{A}))$$

*induced by* $\mathrm{Mod}_{qc}(\mathcal{A}) \subset \mathrm{Mod}(\mathcal{A})$ *is an equivalence.*

We will use $D^b_{qc}(\mathcal{D}_X)$ to denote $D^b(\mathrm{Mod}_{qc}(\mathcal{D}_X))$ since this is the triangulated category of interest to us. The theorem allows us to implicitly identify this with $D^b_{qc}(\mathrm{Mod}(\mathcal{D}_X))$ so there is no conflict with the usual notation.

**Proposition 3.2.2.** *Let $A = \mathcal{D}_X(U)$ for an open affine subset $U \subset X$. Then $A$ is left and right noetherian and has left and right global dimensions $\leq 2 \dim X$.*

*Proof.* Cf. [HTT08, Propositions 1.4.6, D.1.4, Theorem D.2.6][3]. The idea is to reduce everything to the commutative case by using the order filtration on $A$. Let $R = \mathcal{O}_X(U)$, which is a commutative, noetherian, regular ring of global dimension $n$. Since $\Theta_X(U)$ is a locally free $R$-module of rank $n$, we deduce that $\mathrm{gr}^F A \simeq \mathrm{Sym}_R\, \Theta_X(U)$ is a commutative, noetherian, regular ring of global dimension $2n$. We will prove the assertions with respect to left actions. The case of right actions follows by replacing $A$ with $A^{\mathrm{op}}$.

Let $I \subset A$ be a left ideal. Take the filtration on $I$ induced by $F_\bullet A$. Then $\mathrm{gr}^F I \subset \mathrm{gr}^F A$ is a finitely generated ideal. One sees by induction and $R = F_0 A$ that the generators of $\mathrm{gr}^F I$ lift to generators of $I$. Hence $A$ is left noetherian.

To show that $\mathrm{gl}\dim A \leq 2n$, it suffices by a standard argument [Bou07, §8.3] to prove that $\mathrm{Ext}^{2n+1}_A(M, N) = 0$ for finitely generated left $A$-modules $M, N$. Give $M$ a filtration so that $\mathrm{gr}\, M$ is a finitely generated $\mathrm{gr}\, A$-module. Let $C_\bullet \to \mathrm{gr}\, M$ be a resolution by graded free $\mathrm{gr}\, A$-modules of finite rank. This lifts to a resolution $P_\bullet \to M$ of $M$ by filtered free $A$-modules

[3]The proof is more cleanly presented in the language of coherent modules and good filtrations, but to keep the logical flow of our exposition, we give a proof without this terminology. The arguments are the same.

of finite rank such that $\operatorname{gr} P_\bullet = C_\bullet$ (here finite generation is needed to ensure our filtrations are exhaustive). So $P_i$ is a direct sum of copies of $A$ with appropriate shifts in the indices of the filtrations. Put an arbitrary filtration on $N$ so we can consider $H^{2n+1}\operatorname{Hom}_{\operatorname{gr} A}(C_\bullet, \operatorname{gr} N) = \operatorname{Ext}^{2n+1}_{\operatorname{gr} A}(\operatorname{gr} M, \operatorname{gr} N) = 0$. The filtrations on $P_\bullet$ and $N$ induce filtrations of $\operatorname{Hom}_A(P_\bullet, N)$ compatible with the differentials. Since the $P_\bullet$ are free, the associated graded complex is just $\operatorname{Hom}_{\operatorname{gr} A}(C_\bullet, \operatorname{gr} N)$. A filtered complex is acyclic if its associated graded is, so we conclude that $H^{2n+1}\operatorname{Hom}_A(P_\bullet, N) = \operatorname{Ext}^{2n+1}_A(M, N) = 0$. □

**Proposition 3.2.3.** *The category* $\operatorname{Mod}_{qc}(\mathcal{D}_X)$ *has enough flasque injectives, and its global dimension is finite. If $X$ is quasi-projective, then* $\operatorname{Mod}_{qc}(\mathcal{D}_X)$ *has enough locally free objects.*

*Proof.* Cf. [HTT08, Propositions 1.4.14, 1.4.18] and [BGK+87, VI, Theorem 1.10, §2]. □

We remind the reader that whenever we talk about derived categories of $\mathcal{D}$-modules, we will assume the underlying scheme is smooth, quasi-projective, and of pure dimension. As a corollary of the above propositions, any object of $\operatorname{Mod}_{qc}(\mathcal{D}_X)$ admits a bounded injective right resolution and a bounded locally projective left resolution. This will allow us to define the necessary derived functors.

*Remark* 3.2.4. Let $F : \operatorname{Mod}_{qc}(\mathcal{D}_X) \to \operatorname{Mod}_{qc}(\mathcal{D}_Y)$ be a left exact functor. Then the equivalences $D^b_{qc}(\mathcal{D}) \simeq D^b_{qc}(\operatorname{Mod}(\mathcal{D}))$ for $\mathcal{D} = \mathcal{D}_X$ and $\mathcal{D}_Y$ from Theorem 3.2.1 together with the universal property of derived functors allows us to compute the right derived functor $RF$ using a complex of injective objects in either $\operatorname{Mod}_{qc}(\mathcal{D}_X)$ or $\operatorname{Mod}(\mathcal{D}_X)$.

3.2.5. Since $\overrightarrow{\Omega}, \overleftarrow{\Omega}$ are exact, we have induced quasi-inverse functors

$$\overrightarrow{\Omega} : D^b_{qc}(\mathcal{D}_X) \rightleftarrows D^b_{qc}(\mathcal{D}_X^{\mathrm{op}}) : \overleftarrow{\Omega}.$$

## 3.3. **Pullback and pushforward.** Let $\varphi : Y \to X$ be a morphism of schemes.

3.3.1. *Pullback.* In 2.2.7 we defined the pullback functor $\varphi^* : \operatorname{Mod}_{qc}(\mathcal{D}_X) \to \operatorname{Mod}_{qc}(\varphi^\sharp\mathcal{D}_X)$. Example 2.2.4 shows that $\varphi^\sharp\mathcal{D}_X \simeq \mathcal{D}_Y$, so $\varphi^*$ defines the pullback from $\mathcal{D}_X$-modules to $\mathcal{D}_Y$-modules, and this functor coincides with the inverse image of $\mathcal{O}$-modules. Unravelling the constructions (see (2.2.1.1)), we see that the $\Theta_Y$-action on $\varphi^*\mathcal{M} = \mathcal{O}_Y \otimes_{\varphi^{-1}\mathcal{O}_X} \varphi^{-1}\mathcal{M}$ for a $\mathcal{D}_X$-module $\mathcal{M}$ is given by

$$\xi(f \otimes m) = \xi(f) \otimes m + f d\varphi(\xi)(m) \qquad (\xi \in \Theta_Y,\ f \in \mathcal{O}_Y,\ m \in \mathcal{M})$$

where $d\varphi : \Theta_Y \to \varphi^*\Theta_X$ is the differential of $\varphi$. We remark that $\varphi^*$ is naturally a tensor functor.

3.3.2. It is more convenient to give another description of the pullback. Let $\mathcal{D}_{Y\to X}$ denote the $(\mathcal{D}_Y, \varphi^{-1}\mathcal{D}_X)$-bimodule $\varphi^*\mathcal{D}_X$. Then by definition

$$\varphi^*\mathcal{M} = \mathcal{D}_{Y\to X} \underset{\varphi^{-1}\mathcal{D}_X}{\otimes} \varphi^{-1}\mathcal{M}.$$

3.3.3. Since objects of $\operatorname{Mod}_{qc}(\mathcal{D}_X)$ admit bounded locally projective (hence flat) resolutions, we have the left derived functor $L\varphi^* : D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_Y)$, which is given by

$$L\varphi^*(\mathcal{M}^\bullet) = \mathcal{D}_{Y\to X} \underset{\varphi^{-1}\mathcal{D}_X}{\overset{L}{\otimes}} \varphi^{-1}(\mathcal{M}^\bullet)$$

since $\varphi^{-1}$ is exact. It turns out that it will be convenient to consider the shifted functor $\varphi^! := L\varphi^*[\dim Y - \dim X] : D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_Y)$.

3.3.4. *Pushforward.* It is more natural to define pushforwards for *right* $\mathcal{D}$-modules. We define the functor $\varphi_\star : D^b(\mathcal{D}_Y^{\mathrm{op}}) \to D^b(\mathcal{D}_X^{\mathrm{op}})$ by

$$\varphi_\star(\mathcal{M}'^\bullet) = R\varphi_\bullet\Big(\mathcal{M}'^\bullet \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{D}_{Y\to X}\Big).$$

Here $\bullet \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ sends $D^b(\mathcal{D}_Y^{\mathrm{op}}) \to D^b(\varphi^{-1}(\mathcal{D}_X^{\mathrm{op}}))$ and $R\varphi_\bullet : D(\varphi^{-1}\mathcal{D}_X^{\mathrm{op}}) \to D(\mathcal{D}_X^{\mathrm{op}})$ is the right derived functor of the direct image functor $\varphi_\bullet$ for sheaves. Since $Y$ is noetherian of finite dimension, it follows from sheaf theory (cf. [HTT08, Proposition 1.5.4], [Har77, III, Theorem 2.7]) that $R\varphi_\bullet$ preserves boundedness. Note that it is not clear that $\varphi_\star$ preserves quasi-coherence. This will be shown later in Proposition 3.6.4.

*Remark* 3.3.5. The definition of $\varphi_\star$ involves both a right exact functor $\bullet \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ and a left exact functor $\varphi_\bullet$. Without the use of derived categories, this definition does not make much sense. In particular, it is not clear if quasi-coherence is preserved, and the composition rule (see Proposition 3.3.7) may not hold.

We prefer to work with left $\mathcal{D}$-modules, so let us define pushforward in this case using the side changing operators. For $\mathcal{M}^\bullet \in D^b(\mathcal{D}_Y)$, put $\mathcal{F}'^\bullet = \overrightarrow{\Omega}(\mathcal{M}^\bullet) \otimes^L_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$. We have the following projection formula relating $\mathcal{D}$-modules:

**Lemma 3.3.6.** *For $\mathcal{F}'^\bullet \in D^b(\varphi^{-1}(\mathcal{D}_X^{\mathrm{op}}))$ and $\mathcal{N}^\bullet \in D^b_{qc}(\mathcal{D}_X)$, there is a canonical isomorphism*

$$R\varphi_\bullet(\mathcal{F}'^\bullet) \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{N}^\bullet \xrightarrow{\sim} R\varphi_\bullet\Big(\mathcal{F}'^\bullet \overset{L}{\underset{\varphi^{-1}\mathcal{D}_X}{\otimes}} \varphi^{-1}(\mathcal{N}^\bullet)\Big)$$

*in $D^b(\mathrm{Sh}(X))$, where $\mathrm{Sh}(X)$ is the category of abelian sheaves on $X$.*

*Proof.* The morphism is defined using the adjunction of $R\varphi_\bullet, \varphi^{-1}$ and the compatibility of $\varphi^{-1}$ with tensor products. The assertion is local, so we assume $X$ is affine. Since $X$ is noetherian, $R\varphi_\bullet$ commutes with direct limits [Har77, III, Proposition 2.9], so by taking the homotopy limit of truncations, we may assume $\mathcal{N}$ is a single $\mathcal{D}_X$-module. Moreover $X$ is affine, so we can replace $\mathcal{N}$ by a free resolution. The morphism is clearly an isomorphism in this case. ☐

Since ${}^\Omega\mathcal{D}_Y$ and $\mathcal{D}_X^\Omega$ are locally free, Lemma 3.3.6 and associativity of (derived) tensor products gives a canonical isomorphism

$$\overleftarrow{\Omega}\varphi_\star\overrightarrow{\Omega}(\mathcal{M}^\bullet) = R\varphi_\bullet(\mathcal{F}'^\bullet) \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_X^\Omega \xrightarrow{\sim} R\varphi_\bullet\Big(\big({}^\Omega\mathcal{D}_Y \underset{\mathcal{D}_Y}{\otimes} \mathcal{D}_{Y\to X} \underset{\varphi^{-1}\mathcal{D}_X}{\otimes} \varphi^{-1}(\mathcal{D}_X^\Omega)\big) \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{M}^\bullet\Big).$$

Defining the $(\varphi^{-1}(\mathcal{D}_X), \mathcal{D}_Y)$-bimodule $\mathcal{D}_{X\leftarrow Y} = \overrightarrow{\Omega}(\varphi^*(\mathcal{D}_X^\Omega))$, we have shown that the pushforward on left $\mathcal{D}$-modules $\varphi_\star : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ can be equivalently defined by

$$\varphi_\star(\mathcal{M}^\bullet) = R\varphi_\bullet\Big(\mathcal{D}_{X\leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{M}^\bullet\Big).$$

**Proposition 3.3.7.** *Let $Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$ be morphisms of schemes. Then we have canonical isomorphisms $L\psi^*L\varphi^* \xrightarrow{\sim} L(\varphi\circ\psi)^*$ and $\varphi_\star\psi_\star \xrightarrow{\sim} (\varphi\circ\psi)_\star$.*

*Proof.* We have a canonical isomorphism $\psi^*\varphi^* \simeq (\varphi\circ\psi)^*$ of $\mathcal{D}_Z$-modules (see 2.2.8). Let $\mathcal{M}^\bullet$ be a bounded complex of locally projective $\mathcal{D}_X$-modules, so $L\varphi^*(\mathcal{M}^\bullet) = \varphi^*(\mathcal{M}^\bullet)$. Take a bounded locally projective resolution $\mathcal{P}^\bullet \to \varphi^*(\mathcal{M}^\bullet)$ in $D^b_{qc}(\mathcal{D}_Y)$. Then $L\psi^*L\varphi^*(\mathcal{M}^\bullet) = \psi^*(\mathcal{P}^\bullet)$, and $\psi^*(\mathcal{P}^\bullet) \to \psi^*\varphi^*(\mathcal{M}^\bullet) \simeq (\varphi\circ\psi)^*(\mathcal{M}^\bullet) = L(\varphi\circ\psi)^*(\mathcal{M}^\bullet)$ defines the canonical morphism $L\psi^*L\varphi^* \to L(\varphi\circ\psi)^*$ by the universal property of derived functors. Since $\mathcal{D}$ is $\mathcal{O}$-flat, both $\mathcal{P}^\bullet$ and $\varphi^*(\mathcal{M}^\bullet)$ consist of flat $\mathcal{O}_Y$-modules. Therefore by applying the forgetful functor to $\mathcal{O}$-modules, we deduce that $\psi^*(\mathcal{P}^\bullet) \to \psi^*\varphi^*(\mathcal{M}^\bullet)$ is an isomorphism in $D^b_{qc}(\mathcal{D}_Z)$.

We will show composition of pushforwards for right $\mathcal{D}$-modules. Take $\mathcal{M}'^{\bullet} \in D^b(\mathcal{D}_Z^{\mathrm{op}})$. Lemma 3.3.6 gives us the canonical isomorphism

$$R\varphi_\bullet\Big(R\psi_\bullet(\mathcal{M}'^{\bullet} \overset{L}{\underset{\mathcal{D}_Z}{\otimes}} \mathcal{D}_{Z\to Y}) \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{D}_{Y\to X}\Big) \xrightarrow{\sim} R\varphi_\bullet R\psi_\bullet\Big((\mathcal{M}'^{\bullet} \overset{L}{\underset{\mathcal{D}_Z}{\otimes}} \mathcal{D}_{Z\to Y}) \overset{L}{\underset{\varphi^{-1}\mathcal{D}_Y}{\otimes}} \varphi^{-1}\mathcal{D}_{Y\to X}\Big)$$

which one sees is compatible with the $\mathcal{D}_X$-action. The usual composition rule for derived direct images of sheaves and associativity of derived tensor products [KS94, Exercise II.24] imply that the RHS is isomorphic to $R(\varphi\circ\psi)_\bullet(\mathcal{M}'^{\bullet} \otimes^L_{\mathcal{D}_Z} L\psi^*(\mathcal{D}_{Y\to X}))$. Since $\mathcal{D}_X$ is $\mathcal{O}_X$-flat, $\mathcal{D}_{Y\to X} = L\varphi^*\mathcal{D}_X$. Thus the composition rule for pullbacks gives $L\psi^*(\mathcal{D}_{Y\to X}) \simeq \mathcal{D}_{Z\to X}$ in $D(\mathcal{D}_Z)$. We conclude that the RHS is isomorphic to $(\varphi\circ\psi)_\star(\mathcal{M}'^{\bullet})$. □

3.4. **Closed embeddings and Kashiwara's theorem.** By considering the graph, any morphism between schemes can be factored as the composition of a locally closed embedding and a projection. In the following two subsections we will look more closely at what the pullback and pushforward functors do in each of these situations.

3.4.1. *Open embeddings.* Let $j : U \hookrightarrow X$ be an open embedding. Then $j^{-1} = j^* = j^!$ and $\mathcal{D}_{U\hookrightarrow X} = \mathcal{D}_X|_U = \mathcal{D}_U$ as a bimodule with the usual left and right actions. It follows that $\mathcal{D}_{X\hookleftarrow U} = \mathcal{D}_U$ as well. Therefore $j_\star = Rj_\bullet$ coincides with the derived sheaf direct image.

Since $X$ is noetherian, $j$ is quasi-compact so $j_\bullet$ preserves quasi-coherence. Recall that $j^*$ is left adjoint to $j_\bullet$ and $j^*j_\bullet = \mathrm{id}_U$. For an arbitrary $\mathcal{D}_X$-module $\mathcal{M}$, the kernel and cokernel of the morphism $\mathcal{M} \to j_\bullet j^*\mathcal{M}$ are supported on the closed subset $Z = X - U$ (we do not give $Z$ a scheme structure). Let us consider the left exact functor $\Gamma_Z : \mathrm{Mod}_{qc}(\mathcal{D}_X) \to \mathrm{Mod}_{qc}(\mathcal{D}_X)$ given by taking the kernel $\Gamma_Z(\mathcal{M}) := \ker(\mathcal{F} \to j_\bullet j^*\mathcal{F})$. For an open $V \subset X$, the local sections $\Gamma(V, \Gamma_Z(\mathcal{M})) = \{m \in \mathcal{M}(V) \mid \mathrm{supp}(m) \subset Z\}$. If $\mathcal{M}$ is flasque, then $\mathcal{M} \twoheadrightarrow j_\bullet j^*(\mathcal{M})$ is surjective. Since $\mathrm{Mod}_{qc}(\mathcal{D}_X)$ has enough flasque injectives (Proposition 3.2.3), we always have a distinguished triangle

$$R\Gamma_Z(\mathcal{M}^\bullet) \to \mathcal{M}^\bullet \to j_\star j^!(\mathcal{M}^\bullet) \tag{3.4.1.1}$$

for any $\mathcal{M}^\bullet \in D^b_{qc}(\mathcal{D}_X)$.

3.4.2. Let $\mathrm{Mod}_{qc,Z}(\mathcal{D}_X) \subset \mathrm{Mod}_{qc}(\mathcal{D}_X)$ denote the full subcategory of modules which are set-theoretically supported on $Z$, and let $D^b_{qc,Z}(\mathcal{D}_X)$ denote the full subcategory of $D^b_{qc}(\mathcal{D}_X)$ consisting of complexes with cohomology set-theoretically supported on $Z$. We deduce from (3.4.1.1) that $R\Gamma_Z$ defines a quasi-inverse to the inclusion

$$D^b(\mathrm{Mod}_{qc,Z}(\mathcal{D}_X)) \to D^b_{qc,Z}(\mathcal{D}_X),$$

so we in fact have an equivalence of categories.

3.4.3. *Closed embeddings.* Let $i : Y \hookrightarrow X$ be a closed embedding of schemes, with defining ideal $\mathcal{I} \subset \mathcal{O}_X$. Let $r = \dim Y$ and $n = \dim X$.

**Example 3.4.4.** To really see what's going on, first assume that $X$ is affine with a coordinate system $(x_j, \partial_j)_{j\le n}$ such that $\mathcal{I} = (x_{r+1}, \ldots, x_n)$ and $(x_j \circ i, \partial_i)_{j\le r}$ form a coordinate system for $X$ (here for $j \le r$ we use $\partial_j$ to denote both the derivation of $\mathcal{O}_X$ and the induced derivation of $\mathcal{O}_Y$). Let's see what $\mathcal{D}_{Y\to X}$ and $\mathcal{D}_{X\leftarrow Y}$ look like in this situation (cf. [HTT08, Examples 1.3.2, 1.3.5, 1.5.23]).

Recall that $\mathcal{D}_X \simeq \mathcal{O}_X[\partial_1, \ldots, \partial_n]$ as left $\mathcal{O}_X$-modules. As we are in the situation where the tangent space $\Theta_X$ splits, we have a subring $\mathcal{D}' = \mathcal{O}_X[\partial_1, \ldots, \partial_r] \subset \mathcal{D}_X$ and $\mathcal{D}_X \simeq \mathcal{D}' \otimes_k k[\partial_{r+1}, \ldots, \partial_n]$. Observing that $\mathcal{D}_Y \simeq i^*\mathcal{D}'$, we see that $\mathcal{D}_{Y\to X} = i^*\mathcal{D}_X$ is isomorphic to $\mathcal{D}_Y \otimes_k k[\partial_{r+1}, \ldots, \partial_n]$ as left $\mathcal{D}_Y$-modules. The right $i^{-1}\mathcal{D}_X$-action is evident.

Let $y_j = x_j \circ i$ for $j \le r$. Then $dx_1 \cdots dx_n$ and $dy_1 \cdots dy_r$ simultaneously trivialize $\Omega_X^n$ and $\Omega_Y^r$ respectively, so $\mathcal{D}_{X \leftarrow Y}$ is isomorphic *as a sheaf of vector spaces* to $\mathcal{D}_{Y \to X} \simeq \mathcal{O}_Y[\partial_1, \dots, \partial_n]$. We use Example 3.1.9 to describe the $(i^{-1}\mathcal{D}_X, \mathcal{D}_Y)$-actions. Take $m \in \mathcal{D}_{X \leftarrow Y}$, $P \in \mathcal{D}_Y$, and $Q \in i^{-1}\mathcal{D}_X$. Then $mP := ({}^tP)m$ and $Qm := m({}^tQ)$, where the actions on the RHS are the familiar ones in $\mathcal{D}_{Y \to X}$.

Observe from our explicit description that $\mathcal{D}_{X \leftarrow Y} \simeq k[\partial_1, \dots, \partial_n] \otimes_k \mathcal{O}_Y$ as an $i^{-1}\mathcal{O}_X$-module, where $i^{-1}\mathcal{O}_X$ acts via the surjection to $\mathcal{O}_Y$. Consequently, $i_\bullet \mathcal{D}_{X \leftarrow Y}$ is a quasi-coherent $\mathcal{O}_X$-module since $i_\bullet \mathcal{O}_Y$ is.

We now return to the general case. The smoothness of $X$ and $Y$ imply that $\Theta_Y \hookrightarrow i^*\Theta_X$ splits locally, so for every point $y \in Y$ there is a neighborhood $U \subset X$ containing $y$ satisfying the hypotheses of Example 3.4.4. From the local case we deduce that $\mathcal{D}_{X \leftarrow Y}$ is locally free of infinite rank (unless $r = n$) as a $\mathcal{D}_Y$-module. By considering locally free resolutions in $D^b_{qc}(\mathcal{D}_Y)$, we deduce that $i_\star$ preserves quasi-coherence since $i_\bullet \mathcal{D}_{X \leftarrow Y}$ is quasi-coherent.

On the other hand, $i$ is affine, so $i_\bullet$ is exact. Therefore $i_\star \simeq i_+$ where $i_+ : \mathrm{Mod}_{qc}(\mathcal{D}_Y) \to \mathrm{Mod}_{qc}(\mathcal{D}_X)$ is the exact functor given by $i_+(\mathcal{M}) = i_\bullet(\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{M})$. Define the left exact functor $i^+ : \mathrm{Mod}_{qc}(\mathcal{D}_X) \to \mathrm{Mod}_{qc}(\mathcal{D}_Y)$ by

$$i^+(\mathcal{N}) = \mathcal{H}om_{i^{-1}(\mathcal{D}_X)}(\mathcal{D}_{X \leftarrow Y}, i^{-1}(\mathcal{N})).$$

To see that $i^+$ preserves quasi-coherence, let us reduce to the local case. The discussion of Example 3.4.4 implies that $i^+(\mathcal{N}) \simeq \mathcal{H}om_{i^{-1}\mathcal{O}_X}(\mathcal{O}_Y, i^{-1}\mathcal{N})$, which is the quasi-coherent $\mathcal{O}_X$-module corresponding to the sections of $\mathcal{N}$ killed by $\mathcal{I}$.

**Lemma 3.4.5.** *The functor $i_+$ is left adjoint to $i^+$.*

*Proof.* Cf. [HTT08, Proposition 1.5.25]. Recall that $i_\bullet : \mathrm{Mod}(i^{-1}\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_Y)$ is fully faithful. We first show that the natural map

$$\mathcal{H}om_{\mathcal{D}_X}\Big(i_\bullet(\mathcal{D}_{X \leftarrow Y} \underset{\mathcal{D}_Y}{\otimes} \mathcal{M}), \mathcal{N}\Big) \to i_\bullet \mathcal{H}om_{i^{-1}\mathcal{D}_X}\Big(\mathcal{D}_{X \leftarrow Y} \underset{\mathcal{D}_Y}{\otimes} \mathcal{M}, i^{-1}\mathcal{N}\Big)$$

induced by $\mathcal{N} \to i_\bullet i^{-1}\mathcal{N}$ is an isomorphism. The assertion is local, so assume the situation of Example 3.4.4. Then $\mathcal{D}_{X \leftarrow Y} \simeq k[\partial_{r+1}, \dots, \partial_n] \otimes_k \mathcal{D}_Y$ as a right $\mathcal{D}_Y$-module. Take a morphism $\phi : \mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{M} \to i^{-1}\mathcal{N}$ of $i^{-1}\mathcal{D}_X$-modules. For $f \in i^{-1}\mathcal{I}$ and $m \in \mathcal{M}$, one sees that

$$f \cdot \phi(1 \otimes m) = \phi(1 \otimes fm) = 0.$$

The image of $\phi$ is generated by $\phi(1 \otimes m)$ as an $i^{-1}\mathcal{D}_X$-module, so $\phi$ factors through $i^{-1}\Gamma_Y(\mathcal{N})$. Noting that $\Gamma_Y(\mathcal{N}) \simeq i_\bullet i^{-1}\Gamma_Y(\mathcal{N})$ since $\Gamma_Y(\mathcal{N}) \subset \mathcal{N}$ lies in the essential image of $i_\bullet$, we get the desired inverse map $i_\bullet(\phi)$.

To finish, apply the adjunction of $\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \bullet$ and $\mathcal{H}om_{i^{-1}\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Y}, \bullet)$ to the RHS; then take global sections on $X$. $\square$

**Theorem 3.4.6** (Kashiwara)**.** *The functors $i_+ : \mathrm{Mod}_{qc}(\mathcal{D}_Y) \rightleftarrows \mathrm{Mod}_{qc,Y}(\mathcal{D}_X) : i^+$ are quasi-inverse and define an equivalence of categories.*

*Proof.* It suffices to show that the unit and counit of adjunction are isomorphisms, so we may work locally. Since $i_+ = i_\star$, Proposition 3.3.7 implies that $i_+$ is compatible with composition. Hence by induction we reduce to the setting of Example 3.4.4 with $r = n - 1$. Put $x = x_n$ and $\partial = \partial_n$. Then $\mathcal{D}_{X \leftarrow Y} \simeq k[\partial] \otimes_k \mathcal{D}_Y$ with actions via formal adjunctions.

Consider the unit $\mathrm{id} \to i^+ i_+$. For $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_Y)$,

$$i^+ i_+(\mathcal{M}) \simeq \mathcal{H}om_{i^{-1}\mathcal{D}_X}\Big(\mathcal{O}_Y \underset{i^{-1}\mathcal{O}_X}{\otimes} i^{-1}\mathcal{D}_X, k[\partial] \underset{k}{\otimes} \mathcal{M}\Big).$$

Since $\mathcal{O}_Y = i^{-1}\mathcal{O}_X/(x)$, the RHS is the kernel of the left action of $x$ on $k[\partial]\otimes_k \mathcal{M}$. Calculating, $\partial^j x = x\partial^j + j\partial^{j-1} \in \mathcal{D}_X$. Thus from the twisted actions on $\mathcal{D}_{X\leftarrow Y}$ we see that $x(\partial^j \otimes m) = \partial^j \otimes (xm) + j\partial^{j-1}\otimes m = j\partial^{j-1}\otimes m$. Therefore the kernel is just $1 \otimes \mathcal{M}$, as desired.

Now we show $i_+ i^+ \to \mathrm{id}$ is an isomorphism. Take $\mathcal{N} \in \mathrm{Mod}_{qc,Y}(\mathcal{D}_X)$. Consider the operator $\theta = x\partial \in \mathcal{D}_X$ and the $\theta$-eigenspaces $\mathcal{N}^j = \{m \mid \theta m = jm\}$ for $j \in \mathbf{Z}$. Using $\partial x = \theta + 1$, we have

$$x : \mathcal{N}^j \rightleftarrows \mathcal{N}^{j+1} : \partial.$$

Since $\theta : \mathcal{N}^j \to \mathcal{N}^j$ is an isomorphism for $j < 0$ and $\partial x : \mathcal{N}^j \to \mathcal{N}^j$ is an isomorphism for $j < -1$, we get that $x : \mathcal{N}^j \to \mathcal{N}^{j+1}$ and $\partial : \mathcal{N}^{j+1} \to \mathcal{N}^j$ are isomorphisms for $j < -1$. Let us prove that

$$\ker(x^j) \subset \mathcal{N}^{-1} \oplus \cdots \oplus \mathcal{N}^{-j}$$

by induction on $j \geq 1$. If $m \in \ker(x)$, then $\partial x m = (\theta+1)m = 0$ implies $m \in \mathcal{N}^{-1}$. For $j > 1$, assume the claim for $j-1$ and take $m \in \ker(x^j)$. Then $xm \in \mathcal{N}^{-1}\oplus\cdots\oplus\mathcal{N}^{-j+1}$ implies $\partial x m \in \mathcal{N}^{-2}\oplus\cdots\oplus\mathcal{N}^{-j}$. On the other hand, $x^{j-1}(\theta m + jm) = 0$. Therefore again by the inductive hypothesis, $\theta m + jm \in \mathcal{N}^{-1}\oplus\cdots\oplus\mathcal{N}^{-j+1}$. Taking the difference, we find that $\theta m - \partial x m + jm = (j-1)m \in \mathcal{N}^{-1}\oplus\cdots\oplus\mathcal{N}^{-j}$.

Since $\mathcal{N}$ is set-theoretically supported on $Y$ and quasi-coherent, for any $m \in \mathcal{N}$ there exists $j$ such that $x^j m = 0$. Therefore $\mathcal{N} = k[\partial]\otimes_k \mathcal{N}^{-1}$. It follows from the decomposition that $\ker(x) = \mathcal{N}^{-1}$. From an earlier discussion, $i_\bullet i^+(\mathcal{N}) \simeq \ker(x)$, so we win. ☐

3.4.7. We now consider what happens in the derived category.

**Proposition 3.4.8.** *There is a functorial isomorphism $Ri^+ \simeq i^! : D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_Y)$.*

*Proof.* Cf. [BGK+87, VI, §7] and [HTT08, Proposition 1.5.16]. We first prove the claim at the cohomological level, i.e., that for $\mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{D}_X)$, there exist functorial isomorphisms of $\mathcal{D}_Y$-modules $R^p i^+(\mathcal{N}) \simeq L^{n-r-p} i^*(\mathcal{N})$ for all $p$. Suppose for the moment that we are in the local situation of Example 3.4.4. We use the same notation. Then $Y \hookrightarrow X$ is a regular embedding, so we have the *Koszul resolution*

$$0 \to \overset{n-r}{\wedge} V \underset{k}{\otimes} i^{-1}\mathcal{O}_X \to \cdots \to V \underset{k}{\otimes} i^{-1}\mathcal{O}_X \to i^{-1}\mathcal{O}_X \to \mathcal{O}_Y \to 0$$

where $V$ is the $k$-vector space with basis $dx_{r+1}, \ldots, dx_n$. The differential is given by

$$dx_{j_1}\wedge\cdots\wedge dx_{j_p} \mapsto \sum_{q=1}^{s} (-1)^{q+1} x_{j_q} dx_{j_1}\wedge\cdots\wedge \widehat{dx_{j_q}} \wedge\cdots\wedge dx_{j_p}.$$

We tensor on the right by $i^{-1}\mathcal{D}_X$ to get a resolution $\wedge^\bullet V \otimes_k i^{-1}\mathcal{D}_X \to \mathcal{D}_{Y\to X}$. Recall that there is a subring $\mathcal{D}' \simeq \mathcal{O}_X[\partial_1, \ldots, \partial_r] \subset \mathcal{D}_X$ commuting with $x_j$ for $j > r$. Therefore the above is a resolution of $(i^{-1}\mathcal{D}', i^{-1}\mathcal{D}_X)$-bimodules. By choosing the trivializations of $\Omega^n_X$ and $\Omega^r_Y$ by $dx_1\cdots dx_n$ and $dy_1\cdots dy_r$ respectively, side changing via formal adjunction gives us a resolution $\wedge^\bullet V \otimes_k i^{-1}\mathcal{D}_X \to \mathcal{D}_{X\leftarrow Y}$ of $(\mathcal{D}_X, i^{-1}\mathcal{D}')$-bimodules. The canonical isomorphism

$$\mathcal{H}om_{i^{-1}\mathcal{D}_X}\Big(\overset{p}{\wedge} V \underset{k}{\otimes} i^{-1}\mathcal{D}_X, i^{-1}\mathcal{N}\Big) \xleftarrow{\sim} (\overset{p}{\wedge} V)^* \underset{k}{\otimes} i^{-1}\mathcal{N}$$

respects the left $i^{-1}\mathcal{D}'$-actions (adjunction is an involution). We have the natural pairing $\wedge^{n-r-p} V \xrightarrow{\sim} (\wedge^p V)^* \otimes \wedge^{n-r} V : \omega \mapsto (\omega \wedge \bullet)$. The differential on $\wedge^{n-r-\bullet} V \otimes_k i^{-1}\mathcal{N}$ induced from this pairing is the Koszul differential [Eis95, Proposition 17.15] and commutes with the left $i^{-1}\mathcal{D}'$-action. This provides us the local isomorphism

$$L^{n-r-p} i^*(\mathcal{N}) \simeq \mathcal{E}xt^p_{i^{-1}\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, i^{-1}\mathcal{N}),$$

which is compatible with the $i^{-1}\mathcal{D}'$-action. We have a surjection of rings $i^{-1}\mathcal{D}' \twoheadrightarrow \mathcal{D}_Y$, and both of the above objects live in $\mathrm{Mod}_{qc}(\mathcal{D}_Y)$, so our isomorphism is $\mathcal{D}_Y$-linear. Now one observes that our choices of trivializations are compatible with a change of basis for the $dx_j$, so these local isomorphisms of sheaves glue to give an isomorphism of $\mathcal{D}_Y$-modules in the general case.

Next, we show that

$$\mathcal{E}xt^p_{i^{-1}\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, i^{-1}\mathcal{N}) \simeq R^p i^+(\mathcal{N}).$$

The proof of Lemma 3.4.5 shows that the natural map $\mathcal{H}om_{i^{-1}\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, i^{-1}\Gamma_Y(\mathcal{N})) \to i^+(\mathcal{N})$ is an isomorphism. We have that $i^{-1}\Gamma_Y$ is right adjoint to $i_\bullet : \mathrm{Mod}(i^{-1}\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X)$, which is exact. Therefore $i^{-1}\Gamma_Y$ sends injectives in $\mathrm{Mod}(\mathcal{D}_X)$ to injectives in $\mathrm{Mod}(i^{-1}\mathcal{D}_X)$, which implies (see Remark 3.2.4) that $Ri^+(\mathcal{N}) \simeq Ri^! R\Gamma_Y(\mathcal{N})$. Now by the distinguished triangle (3.4.1.1), it suffices to show that

$$\mathcal{E}xt^p_{i^{-1}\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, i^{-1}j_\star j^!(\mathcal{N})) \simeq L^{n-r-p} i^* j_\star j^!(\mathcal{N}) = 0$$

for all $p$. The equality on the right follows from cohomology and base change for quasi-coherent sheaves (cf. [LH09, Theorem 3.10.3]).

This proves the cohomological part of the proposition. In particular, we have shown $i^+$ has cohomological dimension $n-r$ and $R^{n-r}i^+ \simeq i^*$. Since $\mathrm{Mod}_{qc}(\mathcal{D}_X)$ has enough injectives and locally projectives, and locally projective objects are $i^*$-acyclic, a formal result[4] of derived categories (cf. [Har66, I, Proposition 7.4]) gives a functorial isomorphism between derived functors $Ri^+ \simeq Li^*[r-n] = i^!$. □

Since $i_+$ is exact and $i^+$ is left exact, Lemma 3.4.5 implies that $i_\star : D^b_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_X)$ is left adjoint to $Ri^+ \simeq i^!$. Kashiwara's theorem implies that restricting to $i_\star : D^b_{qc}(\mathcal{D}_Y) \rightleftarrows D^b(\mathrm{Mod}_{qc,Y}(\mathcal{D}_X)) : Ri^+$ gives an equivalence of categories. We also know (see 3.4.2) that the inclusion $D^b(\mathrm{Mod}_{qc,Y}(\mathcal{D}_X)) \to D^b_{qc}(\mathcal{D}_X)$ is an equivalence with quasi-inverse $R\Gamma_Y$. In the course of proving Proposition 3.4.8, we showed that $Ri^+ \simeq Ri^+ R\Gamma_Y : D^b_{qc}(\mathcal{D}_X) \to D^b_{qc}(\mathcal{D}_Y)$. We summarize these results in view of $Ri^+ \simeq i^!$.

**Corollary 3.4.9.** *The functors $i_\star : D^b_{qc}(\mathcal{D}_Y) \rightleftarrows D^b_{qc}(\mathcal{D}_X) : i^!$ form an adjoint pair which induce an equivalence of categories between $D^b_{qc}(\mathcal{D}_Y)$ and $D^b_{qc,Y}(\mathcal{D}_X)$. Furthermore on $D^b_{qc}(\mathcal{D}_X)$ we have an isomorphism $i_\star i^! \simeq R\Gamma_Y$.*

*Remark* 3.4.10 ($\mathcal{D}$-modules on singular schemes)*.* We have so far only considered $\mathcal{D}$-modules on smooth schemes. Suppose $Z$ is a singular scheme of finite type. Then $\mathcal{D}_Z$ is still defined, but it may behave poorly, so we do not want to study modules on it. Instead, we locally embed $Z \hookrightarrow X$ as a closed subscheme of a smooth scheme $X$ and *define* the $\mathcal{D}$-modules on $Z$ to be $\mathrm{Mod}_{qc,Z}(\mathcal{D}_X)$. Kashiwara's theorem implies that this category is well-defined locally. Now we glue these local pieces together to construct a category of $\mathcal{D}$-modules in the general case (see [Gai05, 5.11] for details).

We will continue working with only smooth schemes.

3.5. **Some applications.** Let us give a few examples of how Kashiwara's theorem can be used.

3.5.1. *$\mathcal{O}$-coherent $\mathcal{D}$-modules.* We say that a $\mathcal{D}_X$-module $\mathcal{M}$ is $\mathcal{O}$-coherent if it is coherent as an $\mathcal{O}_X$-module. The following can be proved by elementary means [HTT08, Theorem 1.4.10], but we give a proof following [Ber84] (see also [Gin98, Proposition 3.6.1]) as an application of Kashiwara's theorem.

**Proposition 3.5.2.** *An $\mathcal{O}$-coherent $\mathcal{D}_X$-module $\mathcal{M}$ is locally free as an $\mathcal{O}_X$-module.*

[4]We use this argument because it is not *a priori* clear that the isomorphisms we get from the Koszul resolution exist in $D(\mathcal{D}_Y)$.

*Proof.* It suffices to show that the dimension of the fibers $\dim_k(\mathcal{M} \otimes_{\mathcal{O}_X} \kappa(x))$ is locally constant for $x \in X(k)$. To see this, take a basis of $\mathcal{M} \otimes \kappa(x)$ and lift to local sections $m_i$ of $\mathcal{M}$. By Nakayama's lemma and our hypothesis, there exists a neighborhood $U$ such that the $m_i$ generate $\mathcal{M}|_U$ and are linearly independent in $\mathcal{M} \otimes \kappa(y)$ for all $y \in U$. Now if $\Sigma f_i m_i = 0$ for $f_i \in \mathcal{O}(U)$, then $f_i(y) = 0$ for all $y \in U$. Therefore $f_i = 0$ by Nullstellensatz.

By shrinking $X$, we assume it is irreducible and that there exists an étale morphism $f : X \to \mathbf{A}^n$ (which must be faithfully flat). Now for any two geometric points $x, y \in X(k)$, take a line $\mathbf{A}^1 \hookrightarrow \mathbf{A}^n$ containing $f(x)$ and $f(y)$. Since $\mathbf{A}^1$ is geometrically unibranched, [GD67, Corollaire 18.10.3] implies that $C = X \times_{\mathbf{A}^n} \mathbf{A}^1 \hookrightarrow X$ is a smooth sub-curve containing $x$ and $y$. Now by pulling back $\mathcal{D}$-modules, we can assume that $X$ is a smooth curve.

Suppose the stalk $\mathcal{M}_x$ has torsion for some $x \in X(k)$. Let $i_x : \operatorname{Spec} k \hookrightarrow X$ denote the closed embedding of the point $x$. Then $i_x^+(\mathcal{M}) \neq 0$, and Kashiwara's theorem implies that $(i_x)_+ i_x^+(\mathcal{M}) \hookrightarrow \mathcal{M}$ is injective. This submodule is not coherent since $(i_x)_+(k) = (i_x)_\bullet \mathcal{D}_{X \leftarrow \{x\}}$ is an infinite sum of skyscraper sheaves if $n > 0$. Therefore $\mathcal{M}$ is torsion free, hence locally free, since $X$ is a smooth curve. □

3.5.3. *$\mathcal{D}$-modules on $\mathbf{P}^n$.* Recall the notion of a $\mathcal{D}$-affine scheme introduced in 2.1.16.

**Theorem 3.5.4.** *Projective space $\mathbf{P}^n$ is $\mathcal{D}$-affine.*

*Proof.* We consider the vector space $V = k^{n+1}$ as an affine $k$-scheme with coordinate system $(x_i, \partial_i = \frac{\partial}{\partial x_i})$ for $i = 0, \ldots, n$. Let $V^\circ = V - \{0\} \overset{j}{\hookrightarrow} V$ denote the open punctured subspace; so $V^\circ \overset{\pi}{\to} X := \mathbf{P}(V) = \mathbf{P}^n$ is a principal $\mathbf{G}_m$-bundle. Take $\mathcal{M} \in \operatorname{Mod}_{qc}(\mathcal{D}_X)$. Then $\Gamma(V^\circ, \pi^*\mathcal{M})$ is a $\mathbf{G}_m$-module, i.e., it has a $\mathbf{Z}$-grading

$$\bigoplus_{i \in \mathbf{Z}} \Gamma(X, \mathcal{M} \otimes \mathcal{O}(i)).$$

On the other hand, $j_\bullet \pi^*(\mathcal{M}) \in \operatorname{Mod}_{qc}(\mathcal{D}_V)$. Taking global sections $\Gamma(V, \bullet)$, we see that $\Gamma(V^\circ, \pi^*\mathcal{M})$ is a $\mathcal{D}(V)$-module. Consider the action of $\theta = \Sigma\, x_i \partial_i \in \Theta_V$ on $\pi^*(\mathcal{M})|_{U_{x_i}}$. We have that $\{\frac{x_j}{x_i}\}_{j \neq i}$ forms a coordinate system of $\pi(U_{x_i})$, and $\theta(\frac{x_j}{x_i}) = x_i \partial_i(\frac{x_j}{x_i}) + x_j \partial_j(\frac{x_j}{x_i}) = 0$. So $d\pi(\theta) = 0$, and $\theta$ acts on $\pi^*(\mathcal{M})|_{U_{x_i}} \simeq k[x_i, x_i^{-1}] \otimes_k \mathcal{M}|_{\pi(U_{x_i})}$ as $x_i \frac{d}{dx_i}$. Therefore the $\theta$-eigenspaces of $\Gamma(V^\circ, \pi^*\mathcal{M})$ coincide with the $\mathbf{Z}$-grading: $\theta$ acts on $\Gamma(X, \mathcal{M} \otimes \mathcal{O}(i))$ by $i$.

Let $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ be a short exact sequence in $\operatorname{Mod}_{qc}(\mathcal{D}_X)$. Then we have an exact sequence

$$0 \to j_\bullet \pi^*(\mathcal{M}_1) \to j_\bullet \pi^*(\mathcal{M}_2) \to j_\bullet \pi^*(\mathcal{M}_3) \to R^1 j_\bullet \pi^*(\mathcal{M}_1)$$

since $\pi$ is flat and $j_\bullet$ is left exact. The last term $R^1 j_\bullet \pi^*(\mathcal{M}_1)$ is a $\mathcal{D}_V$-module supported at $0 \in V$, so by Kashiwara's theorem, it is a direct sum of $(i_0)_\bullet(\mathcal{D}_{V \leftarrow \{0\}})$, which looks like $(i_0)_\bullet(k)[\partial_0, \ldots, \partial_n]$ as a skyscraper sheaf. We can rewrite $\theta = \Sigma\, \partial_i x_i - (n+1)$, so the formal adjoint ${}^t\theta = -(n+1) - \Sigma\, x_i \partial_i$, which has eigenvalues $-n - \mathbf{Z}_{\geq 1}$. Therefore

$$0 \to \Gamma(V, j_\bullet \pi^* \mathcal{M}_1)^{\theta=0} \to \Gamma(V, j_\bullet \pi^* \mathcal{M}_2)^{\theta=0} \to \Gamma(V, j_\bullet \pi^* \mathcal{M}_3)^{\theta=0} \to 0$$

is exact, i.e., $\Gamma(X, \bullet)$ is exact.

Take $\mathcal{M} \in \operatorname{Mod}_{qc}(X)$ such that $\Gamma(X, \mathcal{M}) = 0$. This implies that $\theta$ acts as an automorphism of $j_\bullet \pi^* \mathcal{M}$. However multiplication by $x_i$ gives an isomorphism of $\theta$-eigenspaces on $\pi^*(\mathcal{M})|_{U_{x_i}}$, so we deduce that $\pi^*\mathcal{M} = 0$. Therefore $\mathcal{M} = 0$ since $\pi$ is faithfully flat. □

The same proof shows that $\mathbf{P}^n \times X$ is $\mathcal{D}$-affine for any smooth affine scheme $X$, since $V \times X$ is still affine. There is just slightly more notation in that case.

3.5.5. *Grothendieck-Cousin methods.* We present a $\mathcal{D}$-module version of the Cousin complex methods developed by Grothendieck, which we will later apply to the flag variety using the Bruhat decomposition. The ideas here follow [Ber84, 2.4] and [Kem78, §7-10]. The latter presents the theory for general sheaves.

Let $X = Z_0 \supset Z_1 \supset \cdots$ be a decreasing sequence of smooth closed subschemes of $X$. We call this a *smooth stratification of* $X$. Denote $i_\ell : Z_\ell - Z_{\ell-1} \hookrightarrow X$, which factors into an open embedding $j_\ell : Z_\ell - Z_{\ell-1} \hookrightarrow Z_{\ell-1}$ and a closed embedding $\bar{i}_\ell : Z_\ell \hookrightarrow X$. Consider a complex $\mathcal{M}^\bullet \in \mathcal{D}^b_{qc}(\mathcal{D}_X)$. From (3.4.1.1) we get a distinguished triangle

$$R\Gamma_{Z_\ell}(\bar{i}^!_{\ell-1}\mathcal{M}^\bullet) \to \bar{i}^!_{\ell-1}(\mathcal{M}^\bullet) \to j_{\ell\star}j^!_\ell\bar{i}^!_{\ell-1}(\mathcal{M}^\bullet).$$

Corollary 3.4.9 tells us that $R\Gamma_{Z_\ell} \simeq \iota_{\ell\star}\iota^!_\ell$ where $\iota_\ell : Z_\ell \hookrightarrow Z_{\ell-1}$. Applying $\bar{i}_{\ell-1,\star}$ to the above triangle and using composition rules, we then get a distinguished triangle

$$R\Gamma_{Z_\ell}(\mathcal{M}^\bullet) \to R\Gamma_{Z_{\ell-1}}(\mathcal{M}^\bullet) \to i_{\ell-1,\star}i^!_{\ell-1}(\mathcal{M}^\bullet).$$

3.6. **Smooth morphisms and products.** Let $\varphi : Y \to X$ be a smooth morphism of relative dimension $r = \dim Y - \dim X$. Smoothness of $\varphi$ implies that we have short exact sequences

$$0 \to \varphi^*\Omega^1_X \to \Omega^1_Y \to \Omega^1_{Y/X} \to 0 \quad \text{and} \quad 0 \to \Theta_{Y/X} \to \Theta_Y \xrightarrow{d\varphi} \varphi^*\Theta_X \to 0$$

of locally free $\mathcal{O}_Y$-modules.

The *exterior derivatives* $d : \Omega^p_{Y/X} \to \Omega^{p+1}_{Y/X}$ are $k$-linear morphisms uniquely defined [GD67, Définition 16.6.3] by the properties:

(i) $d$ coincides with the usual differential $\mathcal{O}_Y \to \Omega^1_{Y/X}$ for $p = 0$,
(ii) $d^2 = 0$,
(iii) $d(\omega_p \wedge \omega_q) = d\omega_p \wedge \omega_q + (-1)^p\omega_p \wedge d\omega_q$ for $\omega_p \in \Omega^p_{Y/X}$ and $\omega_q \in \Omega^q_{Y/X}$.

For an $\mathcal{O}_Y$-module $\mathcal{M}$, a *connection* is a $k$-linear map

$$\nabla : \mathcal{M} \to \Omega^1_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{M}$$

satisfying $\nabla(fm) = df \otimes m + f\nabla(m)$ for $f \in \mathcal{O}_Y$, $m \in \mathcal{M}$. Dually, this is equivalent to giving a $k$-linear map $\Theta_Y \to \mathcal{E}nd_k(\mathcal{M}) : \xi \mapsto \nabla_\xi$ such that $\nabla_{f\xi}(m) = f\nabla_\xi(m)$ and $\nabla_\xi(fm) = \xi(f)m + f\nabla_\xi(m)$. For an $\mathcal{O}_Y$-module with a connection, define the *relative de Rham differentials* $d : \Omega^p_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M} \to \Omega^{p+1}_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M}$ by the properties:

(i) $d$ coincides with $\mathcal{M} \xrightarrow{\nabla} \Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{M} \to \Omega^1_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M}$ for $p = 0$,
(ii) $d(\omega_p \wedge m_q) = d\omega_p \wedge m_q + (-1)^p\omega_p \wedge dm_q$ for $\omega_p \in \Omega^p_{Y/X}$ and $m_q \in \Omega^q_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M}$, where $d\omega_p$ is the exterior derivative.

Using the properties of the exterior derivative and $\nabla$, one sees that $d(f\omega_p \wedge m_q) = d(\omega_p \wedge fm_q)$, so this new differential is well-defined. Note that $d$ is a map between $\mathcal{O}_Y$-modules, but it is clearly not $\mathcal{O}_Y$-linear.

It is more intuitive to think of $\Omega^p_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{M} \simeq \mathcal{H}om_{\mathcal{O}_Y}(\wedge^p \Theta_{Y/X}, \mathcal{M})$. Under this identification, we can explicitly describe the de Rham differential by

$$\begin{aligned} dm_p(\xi_1, \ldots, \xi_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i-1}\nabla_{\xi_i}(m_p(\xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_{p+1})) + \\ &\sum_{i<j}(-1)^{i+j}m_p([\xi_i, \xi_j], \xi_1, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_{p+1}). \end{aligned}$$

We say that the connection $\nabla$ is *flat* or *integrable* if the *curvature* $\nabla^2 : \mathcal{M} \to \Omega^2_Y \otimes_{\mathcal{O}_Y} \mathcal{M}$, which is just $d^2$ in the non-relative case of $Y/k$, is equal to 0. This is equivalent to requiring that $[\nabla_{\xi_1}, \nabla_{\xi_2}] = \nabla_{[\xi_1,\xi_2]}$. Our remarks in 3.1.2 can now be rephrased as: a left $\mathcal{D}_Y$-module $\mathcal{M}$ is the same as a quasi-coherent $\mathcal{O}_Y$-module with a flat connection.

**Corollary 3.6.1** (of Proposition 3.5.2)**.** *There is an equivalence of categories between $\mathcal{O}$-coherent $\mathcal{D}_Y$-modules and vector bundles on $Y$ with flat connections.*

Let $\mathcal{M}$ be a left $\mathcal{D}_Y$-module. Then flatness of $\nabla$ together with property (ii) of $d$ imply that $d^2$ for all $p$. Therefore we can functorially attach to $\mathcal{M}$ the *relative de Rham complex* of sheaves

$$DR_{Y/X}(\mathcal{M}) = \left(\mathcal{M} \to \Omega^1_{Y/X} \underset{\mathcal{O}_Y}{\otimes} \mathcal{M} \to \cdots \to \Omega^r_{Y/X} \underset{\mathcal{O}_Y}{\otimes} \mathcal{M}\right)$$

living in degrees $[0, r]$. Note that the usual de Rham complex corresponding to the exterior derivative is just $DR_{Y/X}(\mathcal{O}_Y)$. Since $\mathcal{D}_Y$ is a $\mathcal{D}_Y$-bimodule, $DR_{Y/X}(\mathcal{D}_Y)$ is a complex of right $\mathcal{D}_Y$-modules. We see that $DR_{Y/X}(\mathcal{M}) \simeq DR_{Y/X}(\mathcal{D}_Y) \otimes_{\mathcal{D}_Y} \mathcal{M}$ as complexes.

Let $n = \dim X$. By considering determinants, we have an isomorphism of $\mathcal{O}_Y$-modules

$$\Omega^r_{Y/X} \simeq \Omega^{n+r}_Y \underset{\varphi^{-1}\mathcal{O}_X}{\otimes} \varphi^{-1}\Omega^{-n}_X.$$

The inclusion $\varphi^{-1}\mathcal{O}_X \hookrightarrow \varphi^{-1}\mathcal{D}_X$ thus induces an inclusion $\Omega^r_{Y/X} \hookrightarrow \mathcal{D}_{X\leftarrow Y}$ of $\mathcal{O}_Y$-modules. Adjunction then gives us a morphism of right $\mathcal{D}_Y$-modules $\alpha : \Omega^r_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \mathcal{D}_{X\leftarrow Y}$. To get a better picture, we go to the local case:

**Example 3.6.2.** For any point $y \in Y$, smoothness of $\varphi$ ensures we have affine neighborhoods $y \in U \subset Y$ and $\varphi(y) \in V \subset X$ with coordinate systems $(x_j \circ \varphi, y_i; \xi_j, \partial_i)_{j\leq n, i\leq r}$ of $U$ and $(x_j, d\varphi(\xi_j))_{j\leq n}$ of $V$. Hence $dy_i$ form a basis of $\Omega_{Y/X}$ and $\partial_i$ a dual basis of $\Theta_{Y/X}$. Replacing $Y$ and $X$ by $U$ and $V$ respectively, let us see what $\mathcal{D}_{Y\to X}$ and $\mathcal{D}_{X\leftarrow Y}$ looks like.

We have $\mathcal{D}_X \simeq \mathcal{O}_X[\xi_1, \dots, \xi_n]$ as a left $\mathcal{O}_X$-module, so $\mathcal{D}_{Y\to X} \simeq \mathcal{O}_Y[\xi_1, \dots, \xi_n]$ as an $\mathcal{O}_Y$-module. Additionally, $\mathcal{D}_Y \simeq \mathcal{O}_Y[\xi_1, \dots, \xi_n, \partial_1, \dots, \partial_r]$ and $d\varphi(\partial_i) = 0$. The actions are evident.

We use $d(x_j \circ \varphi)$ and $dy_i$ to simultaneously trivialize $\Omega^{n+r}_Y$, $\Omega^n_X$, and $\Omega^r_{Y/X}$. Now $\mathcal{D}_{X\leftarrow Y} \simeq \mathcal{O}_Y[\xi_1, \dots, \xi_n]$ as an $\mathcal{O}_Y$-module, and the $(\varphi^{-1}\mathcal{D}_X, \mathcal{D}_Y)$-actions are defined using formal adjoints. The map $\alpha$ defined above corresponds under these trivializations to the map

$$\mathcal{O}_Y[\xi_1, \dots, \xi_n, \partial_1, \dots, \partial_r] \to \mathcal{O}_Y[\xi_1, \dots, \xi_n] : \Sigma\, f_\alpha \xi^\alpha \mapsto \Sigma(-\xi)^\alpha f_\alpha,\ \sum_{i=1}^{r} \mathcal{D}_Y \partial_i \mapsto 0.$$

In particular, $\alpha$ is surjective. In terms of coordinates, the differential of $DR_{Y/X}(\mathcal{M})$ is given by

$$d(\omega_p \otimes m) = d\omega_p \otimes m + \sum_{i=1}^{r} dy_i \wedge \omega_p \otimes \partial_i m.$$

for $\omega_p \in \Omega^p_{Y/X}$ and $m \in \mathcal{M}$. Observe that $d(dy_1 \cdots \widehat{dy_i} \cdots dy_r \otimes m) = (-1)^{i-1} dy_1 \cdots dy_r \otimes \partial_i m$. Thus $\alpha \circ d = 0$ on $\Omega^{r-1}_{Y/X}$, so $\alpha$ defines a map of complexes $DR_{Y/X}(\mathcal{D}_Y)[r] \to \mathcal{D}_{X\leftarrow Y}$.

We return to the general setting, where our above observations concerning $\alpha$ still hold.

**Lemma 3.6.3.** *We have an isomorphism $\alpha : DR_{Y/X}(\mathcal{D}_Y)[r] \xrightarrow{\sim} \mathcal{D}_{X\leftarrow Y}$ in $D^b_{qc}(\mathcal{D}^{\mathrm{op}}_Y)$.*

*Proof.* We give the de Rham complex $DR_{Y/X}(\mathcal{D}_Y)$ a filtration compatible with the differentials by setting $F_q(\Omega^p_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) = \Omega^p_{Y/X} \otimes_{\mathcal{O}_Y} F_{p+q}\mathcal{D}_Y$. Give $\mathcal{D}_{X\leftarrow Y}$ the filtration induced from the order filtration on $\mathcal{D}_X$. Then $\alpha$ becomes a map of filtered complexes, so it suffices to check that $\mathrm{gr}^F(\alpha)$ is an isomorphism. The corresponding map of associated graded complexes is

$$\Omega^\bullet_{Y/X} \underset{\mathcal{O}_Y}{\otimes} \mathrm{Sym}_{\mathcal{O}_Y}(\Theta_Y) \to \Omega^r_{Y/X} \underset{\mathcal{O}_Y}{\otimes} \mathrm{Sym}_{\mathcal{O}_Y}(\varphi^*\Theta_X).$$

Tensoring this by $\Omega^{-r}_{Y/X}$, we get a Koszul resolution, which is acyclic. $\square$

The de Rham complex allows us to calculate the pushforward of $\mathcal{M}^\bullet \in D^b_{qc}(\mathcal{D}_Y)$ by

$$\varphi_\star(\mathcal{M}^\bullet) \simeq R\varphi_\bullet\Big(DR_{Y/X}(\mathcal{D}_Y) \underset{\mathcal{D}_Y}{\otimes} \mathcal{M}^\bullet\Big)[r],$$

which shows that $\varphi_\star$ preserves quasi-coherence.

**Proposition 3.6.4.** *Let $\varphi : Y \to X$ be a morphism of schemes. Then the pushforward $\varphi_\star$ sends $D^b_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_X)$, i.e., quasi-coherence is preserved.*

*Proof.* By considering the graph, $\varphi$ factors as a closed embedding $Y \hookrightarrow X \times Y$ followed by a projection $X \times Y \to X$. We have shown the pushforward preserves quasi-coherence in each case (the projection is a smooth morphism), so we are done by the composition rule. □

We will henceforth always consider $\varphi_\star$ as a functor $D^b_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_X)$.

3.6.5. *Products and projections.* For general smooth morphisms, $DR_{Y/X}(\mathcal{M})$ is only a complex of $\mathcal{O}_Y$-modules, so it does not give a nice description of the $\varphi^{-1}\mathcal{D}_X$ action on $\mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{M}$. The situation is better when $\varphi$ is a projection $X = X_1 \times X_2 \to X_1$ for schemes $X_i$. Let us look more closely at this case.

3.6.6. *Exterior products.* We have an isomorphism of tdos $\boxtimes_{\mathcal{O}} \mathcal{D}_{X_i} \xrightarrow{\sim} \mathcal{D}_X$ (see Examples 2.1.5(ii) and 2.2.6). Flatness of $\mathcal{O}_X$ over $\boxtimes_k \mathcal{O}_{X_i}$ implies flatness of $\mathcal{D}_X$ over $\boxtimes_k \mathcal{D}_{X_i}$. Given $\mathcal{D}_{X_i}$-modules $\mathcal{M}_i$, we have a $\boxtimes_k \mathcal{D}_{X_i}$-module structure on $\boxtimes_k \mathcal{M}_i$. Define

$$\underset{\mathcal{D}}{\boxtimes} \mathcal{M}_i = \mathcal{D}_X \underset{\boxtimes_k \mathcal{D}_{X_i}}{\otimes} (\underset{k}{\boxtimes} \mathcal{M}_i) \in \mathrm{Mod}_{qc}(\mathcal{D}_X).$$

Observe that $\boxtimes_{\mathcal{D}}$ composed with the forgetful functor coincides with $\boxtimes_{\mathcal{O}}$. Of course for open affine subsets $U_i \subset X_i$, these operations are the obvious ones on $U_1 \times U_2$. An equivalent definition is $\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2 \simeq \mathrm{pr}_1^*\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathrm{pr}_2^*\mathcal{M}_2$ as $\mathcal{D}_X$-modules. Flatness implies that $\bullet \boxtimes_{\mathcal{D}} \bullet$ is exact in both factors, so it extends to a bifunctor $\Pi\, D^b_{qc}(\mathcal{D}_{X_i}) \to D^b_{qc}(\mathcal{D}_X)$ on derived categories. There is an obvious compatibility with (derived) pullbacks. We make analogous definitions for right $\mathcal{D}$-modules.

3.6.7. *Side changing and $\boxtimes$.* We have a splitting of $\mathcal{O}_X$-modules $\oplus\, \mathrm{pr}_i^*(\Omega^1_{X_i}) \simeq \Omega^1_X$. Taking determinants gives $\boxtimes_{\mathcal{O}} \Omega^{n_i}_{X_i} \xrightarrow{\sim} \Omega^n_X$ where $n_i = \dim X_i$. This isomorphism evidently respects the right action by $\boxtimes_{\mathcal{O}} \mathcal{D}_{X_i}$. We deduce that

$$\underset{\mathcal{D}}{\boxtimes} \overrightarrow{\Omega}(\mathcal{M}_i) \simeq \overrightarrow{\Omega}(\underset{\mathcal{D}}{\boxtimes} \mathcal{M}_i)$$

as right $\mathcal{D}_X$-modules. As a consequence, note that $\boxtimes_{\mathcal{D}} {}^{\Omega}\mathcal{D}_{X_i}$ has the structure of either a $(\mathcal{D}_X^{\mathrm{op}}, \boxtimes_k \mathcal{D}_{X_i})$- or $(\boxtimes_k \mathcal{D}^{\mathrm{op}}_{X_i}, \mathcal{D}_X)$-bimodule, depending on which action is used when tensoring by $\mathcal{D}_X$. Similar statements hold for $\overleftarrow{\Omega}$.

3.6.8. We return to looking at the relative de Rham complex with respect to $X \to X_1$. By base change, $\Omega^p_{X/X_1} \simeq \mathcal{O}_{X_1} \boxtimes_{\mathcal{O}} \Omega^p_{X_2}$. Hence

$$DR_{X/X_1}(\mathcal{D}_X) \simeq \mathcal{D}_{X_1} \underset{\mathcal{D}}{\boxtimes} DR_{X_2}(\mathcal{D}_{X_2})$$

as complexes of right $\mathcal{D}_X$-modules. Left multiplication on $\mathcal{D}_{X_1}$ does not affect the differential, so this is also a complex of left $\mathrm{pr}_1^{-1}\mathcal{D}_1$-modules. We also have an isomorphism

$$\mathcal{D}_{X_1 \leftarrow X} = \overrightarrow{\Omega}\mathrm{pr}_1^*(\mathcal{D}^{\Omega}_{X_1}) \simeq \overrightarrow{\Omega}(\mathcal{D}^{\Omega}_{X_1}) \underset{\mathcal{D}}{\boxtimes} \overrightarrow{\Omega}(\mathcal{O}_{X_2}) \simeq \mathcal{D}_{X_1} \underset{\mathcal{D}}{\boxtimes} \Omega^{n_2}_{X_2}$$

of $(\mathrm{pr}^{-1}\mathcal{D}_{X_1}, \mathcal{D}_X)$-bimodules. Our map $\alpha : DR_{X/X_1}(\mathcal{D}_X)[n_2] \to \mathcal{D}_{X_1 \leftarrow X}$ is identity on the $\mathcal{D}_{X_1}$ component, so it is in fact an isomorphism in the derived category $D(\mathrm{pr}_1^{-1}\mathcal{D}_{X_1} \otimes_k \mathcal{D}_X^{\mathrm{op}})$ of bimodules. So in the case of a projection, it is easy to get the $\mathcal{D}_{X_1}$-action on the pushforward using the de Rham complex.

3.6.9. *Pushforward and* $\boxtimes$. Let $\varphi_i : Y_i \to X_i$ be arbitrary morphisms of schemes for $i = 1, 2$, and let $X = \Pi X_i$, $Y = \Pi Y_i$, and $\varphi = \Pi \varphi_i : Y \to X$. Let $n_i = \dim X_i$. Then we have a canonical isomorphism

$$\mathcal{D}_{X\leftarrow Y} \simeq \overrightarrow{\Omega}\varphi^*\Big(\underset{\mathcal{D}}{\boxtimes}\mathcal{D}^{\Omega}_{X_i}\Big) \simeq \underset{\mathcal{D}}{\boxtimes}\overrightarrow{\Omega}\varphi_i^*(\mathcal{D}^{\Omega}_{X_i}) = \underset{\mathcal{D}}{\boxtimes}\mathcal{D}_{X_i\leftarrow Y_i}$$

evidently of $(\boxtimes_k \varphi_i^{-1}(\mathcal{D}_{X_i}), \mathcal{D}_Y)$-bimodules.

**Lemma 3.6.10.** *There is a canonical isomorphism*

$$\underset{\mathcal{D}}{\boxtimes}\varphi_{i\star}(\mathcal{M}_i^\bullet) \xrightarrow{\sim} \varphi_\star\Big(\underset{\mathcal{D}}{\boxtimes}\mathcal{M}_i^\bullet\Big) \in D^b_{qc}(\mathcal{D}_X)$$

*for* $\mathcal{M}_i^\bullet \in D^b_{qc}(\mathcal{D}_{Y_i})$.

*Proof.* First, let us define the desired morphism canonically. By adjunction, this is equivalent to specifying a morphism

$$\underset{k}{\boxtimes}\varphi_i^{-1}R\varphi_{i\bullet}\big(\mathcal{D}_{X_i\leftarrow Y_i}\overset{L}{\underset{\mathcal{D}_{Y_i}}{\otimes}}\mathcal{M}_i^\bullet\big) \to \mathcal{D}_{X\leftarrow Y}\overset{L}{\underset{\boxtimes_k \mathcal{D}_{Y_i}}{\otimes}}(\underset{k}{\boxtimes}\mathcal{M}_i^\bullet).$$

in $D(\boxtimes_k \varphi_i^{-1}\mathcal{D}_{X_i})$. We have counits $\varphi_i^{-1}R\varphi_{i\bullet} \to \mathrm{id}$, so the LHS maps to $\boxtimes_k(\mathcal{D}_{X_i\leftarrow Y_i} \otimes^L_{\mathcal{D}_{Y_i}} \mathcal{M}_i^\bullet)$. By assuming $\mathcal{M}_i^\bullet$ consists of flat $\mathcal{D}_{Y_i}$-modules, we can move tensors around to see that this equals $(\boxtimes_k \mathcal{D}_{X_i\leftarrow Y_i}) \otimes^L_{\boxtimes_k \mathcal{D}_{Y_i}} (\boxtimes_k \mathcal{M}_i^\cdot)$. Therefore our morphism is induced by $\boxtimes_k \mathcal{D}_{X_i\leftarrow Y_i} \to \mathcal{D}_{X\leftarrow Y}$.

From the canonical construction, we see that the morphism is compatible with compositions of $\varphi_i$. Therefore by decomposing $\varphi$ as $Y_1 \times Y_2 \to X_1 \times Y_2 \to X_1 \times X_2$, we reduce to considering the case of a morphism $\varphi \times \mathrm{id} : Y \times T \to X \times T$ given $\varphi : Y \to X$. Take $\mathcal{M}^\bullet \in D^b_{qc}(\mathcal{D}_Y)$ and $\mathcal{N}^\bullet \in D^b_{qc}(\mathcal{D}_T)$. We would like to show that

$$\varphi_\star(\mathcal{M}^\bullet)\underset{\mathcal{D}}{\boxtimes}\mathcal{N}^\bullet \xrightarrow{\sim} (\varphi\times\mathrm{id})_\star(\mathcal{M}^\bullet\underset{\mathcal{D}}{\boxtimes}\mathcal{N}^\bullet).$$

Since $\varphi$ can be further decomposed into a closed embedding and a smooth morphism, we can consider each of these cases separately. Moreover, we may assume $\mathcal{M}^\bullet = \mathcal{M} \in \mathrm{Mod}_{qc}(\mathcal{D}_Y)$ consists of a single object. The assertion is local on $X$. From what we have already shown, in both cases we can explicitly represent $\mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{M}$ by a complex consisting of quasi-coherent sheaves on $Y$. We can then deduce the isomorphism from flat base change formulas for quasi-coherent sheaves (cf. [HTT08, Proposition 1.5.30] for details). □

3.7. **Base change.** It turns out that pushforward and pullback intertwine nicely for *arbitrary* morphisms, as long as our all of our schemes are smooth and quasi-projective (which is not the case for quasi-coherent sheaves).

**Theorem 3.7.1.** *Consider a Cartesian square*

$$\begin{tikzcd} Y_Z \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\varphi}"'] & Y \arrow[d, "\varphi"] \\ Z \arrow[r, "\pi"] & X \end{tikzcd}$$

*and suppose all schemes in the diagram are smooth and quasi-projective. There exists a functorial isomorphism* $\pi^!\varphi_\star \simeq \tilde{\varphi}_\star\tilde{\pi}^! : D^b_{qc}(\mathcal{D}_Y) \to D^b_{qc}(\mathcal{D}_Z)$.

*Proof.* Factor $\pi$ into a closed embedding followed by a projection. If $\pi : Z = X \times T \to X$ is a projection, then $Y_Z = Y \times T$ and $\tilde{\varphi} = \varphi \times \mathrm{id}$, so base change is a special case of Lemma 3.6.10.

Suppose that $\pi = i : Z \hookrightarrow X$ is a closed embedding. Consider the Cartesian squares

$$\begin{tikzcd} Y_Z \arrow[r, hook, "\tilde{i}"] \arrow[d, "\tilde{\varphi}"'] & Y \arrow[d, "\varphi"] & Y_U \arrow[l, hook', "\tilde{j}"'] \arrow[d] \\ Z \arrow[r, hook, "i"] & X & U \arrow[l, hook', "j"'] \end{tikzcd}$$

Since $i^!$ is right adjoint to $i_\star$, we have a canonical morphism $\tilde{\varphi}_\star \tilde{i}^! \to i^! \varphi_\star$. By Kashiwara's theorem, it suffices to show this is an isomorphism after applying $i_\star$. Hence we would like to show $\varphi_\star R\Gamma_{Y_Z} \xrightarrow{\sim} R\Gamma_Z \varphi_\star$. Let $\mathcal{M}^\bullet \in D^b_{qc}(\mathcal{D}_Y)$ and $\mathcal{N}^\bullet := \varphi_\star(\mathcal{M}^\bullet)$. From (3.4.1.1) with respect to $Y_U$ and $U$, we get a morphism of distinguished triangles

$$\begin{tikzcd} \varphi_\star R\Gamma_{Y_Z}(\mathcal{M}^\bullet) \arrow[r] \arrow[d] & \varphi_\star(\mathcal{M}^\bullet) \arrow[r] \arrow[d, equal] & \varphi_\star \tilde{j}_\star \tilde{j}^!(\mathcal{M}^\bullet) \arrow[d, "\sim"] \\ R\Gamma_Z(\mathcal{N}^\bullet) \arrow[r] & \mathcal{N}^\bullet \arrow[r] & j_\star j^!(\mathcal{N}^\bullet) \end{tikzcd}$$

in $D^b_{qc}(\mathcal{D}_X)$. The last arrow is an isomorphism since base change clearly holds for an open subset $U$. Therefore the first arrow must also be an isomorphism. □

*Remark* 3.7.2. The isomorphism in Theorem 3.7.1 is canonical when $\pi$ is either a projection or a closed embedding, but it is not obvious whether it is independent of the choice of decomposition of $\pi$. Unfortunately, we have not resolved this question.

## 4. Coherent and holonomic $\mathcal{D}$-modules

This section is a continuation of §3. As before, all schemes are assumed to be smooth, quasi-projective, and of pure dimension. We give an overview of the theory of coherent and holonomic $\mathcal{D}$-modules. We omit proofs more often here compared to the last section, as we find the relevant expositions in [Ber84], [BGK+87], and [HTT08] to be satisfactory now that we have developed the basic theory.

4.1. **Coherence.** There is a general definition of a coherent module over a sheaf of rings on a topological space, but since we are on a scheme $X$ and we know $\mathcal{D}_X$ is quasi-coherent and locally noetherian, a coherent $\mathcal{D}_X$-module is the same as a quasi-coherent $\mathcal{D}_X$-module locally finitely generated over $\mathcal{D}_X$ (cf. [HTT08, Proposition 1.4.9]).

From the corresponding facts concerning coherent $\mathcal{O}_X$-modules, one deduces the following (cf. [HTT08, Corollary 1.4.17]).

**Lemma 4.1.1.**

(i) *Any $\mathcal{D}_X$-module is a union of coherent $\mathcal{D}_X$-submodules.*
(ii) *A coherent $\mathcal{D}_X$-module is globally generated by a coherent $\mathcal{O}_X$-submodule.*
(iii) *Extension principle: if $\mathcal{M}$ is a $\mathcal{D}_X$-module, $U \subset X$ an open subset, and $\mathcal{N}_U \subset \mathcal{M}|_U$ a coherent $\mathcal{D}_U$-submodule, then there exists a coherent $\mathcal{D}_X$-submodule $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N}|_U = \mathcal{N}_U$.*

Let $\mathrm{Mod}_c(\mathcal{D}_X) \subset \mathrm{Mod}_{qc}(\mathcal{D}_X)$ denote the full abelian subcategory of coherent $\mathcal{D}_X$-modules. The proof of Proposition 3.2.3 shows that any coherent $\mathcal{D}_X$-module is a quotient of a locally free $\mathcal{D}_X$-module of finite rank. Let $D^b_c(\mathcal{D}_X) \subset D^b_{qc}(\mathcal{D}_X)$ be the full triangulated subcategory consisting of those complexes whose cohomology sheaves are coherent $\mathcal{D}_X$-modules. The analogue of Theorem 3.2.1 is true in the coherent setting:

**Proposition 4.1.2.** *The natural morphism $D^b(\mathrm{Mod}_c(\mathcal{D}_X)) \to D^b_c(\mathcal{D}_X)$ is an equivalence of categories.*

*Proof.* We refer the reader to [BGK+87, VI, Proposition 2.11] for the proof. The main idea is that since cohomology is bounded and locally finitely generated, there will only be finitely many cocycles and coboundaries to deal with. The extension principle allows us to go from local to global, so we can replace the objects in our complexes with coherent ones. $\square$

**Proposition 4.1.3.** *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then* $\mathrm{supp}(\mathcal{M})$ *is closed in $X$.*

*Proof.* Let $\mathcal{F} \subset \mathcal{M}$ be a coherent $\mathcal{O}_X$-submodule generating $\mathcal{M}$ over $\mathcal{D}_X$. Take a point $x \notin \mathrm{supp}(\mathcal{M}) \supset \mathrm{supp}(\mathcal{F})$. There is some open affine neighborhood $x \in U$ such that $\mathcal{F}|_U = 0$. But then $\mathcal{M}(U) = \mathcal{D}(U) \cdot \mathcal{F}(U) = 0$, so $\mathcal{M}|_U = 0$. Note that this implies $\mathrm{supp}(\mathcal{M}) = \mathrm{supp}(\mathcal{F})$. $\square$

## 4.2. **Proper pushforward.**

**Proposition 4.2.1.** *Let $\varphi : Y \to X$ be a proper morphism. Then $\varphi_\star D^b_c(\mathcal{D}_Y) \subset D^b_c(\mathcal{D}_X)$.*

*Proof.* We are assuming $Y$ is quasi-projective, so there is a locally closed embedding $Y \hookrightarrow \mathbf{P}^n$. The graph of $\varphi$ decomposes as $Y \hookrightarrow Y \times X \hookrightarrow \mathbf{P}^n \times X \to X$. Properness of $\varphi$ implies that $Y \hookrightarrow \mathbf{P}^n \times X$ is in fact a closed embedding. Therefore it suffices to prove the claim for a closed embedding and a projection $\mathbf{P}^n \times X \to X$. Coherence is local, so we may assume that $X$ is affine. By truncation, we only need to show that $\varphi_\star(\mathcal{M}) \in D^b_c(\mathcal{D}_X)$ for a single coherent $\mathcal{D}_Y$-module $\mathcal{M}$.

Suppose $\varphi = i$ is a closed embedding. Then $Y$ is also affine, so $\mathcal{M}$ is the quotient of a free $\mathcal{D}_Y$-module of finite rank. Since $i_+$ is exact, it suffices to check that $i_+(\mathcal{D}_Y) = i_\bullet \mathcal{D}_{X \leftarrow Y}$ is coherent. From Example 3.4.4, we know that $i_\bullet \mathcal{D}_{X\leftarrow Y}$ is locally isomorphic to $\mathcal{D}_X/\mathcal{D}_X\mathcal{I}$ as a $\mathcal{D}_X$-module, where $\mathcal{I}$ is the defining ideal.

We are left with the case $\varphi : Y = \mathbf{P}^n \times X \to X$. Theorem 3.5.4 established that $Y$ is $\mathcal{D}$-affine. Thus $\mathcal{M}$ admits a bounded resolution by direct summands of free $\mathcal{D}_Y$-modules of finite rank. So we are reduced to proving that $\varphi_\star(\mathcal{D}_Y) \in D^b_c(\mathcal{D}_X)$. Since $\mathcal{D}_Y \simeq \mathcal{D}_{\mathbf{P}^n} \boxtimes_{\mathcal{D}} \mathcal{D}_X$, Lemma 3.6.10 implies that $\varphi_\star(\mathcal{D}_Y) \simeq \pi_\star(\mathcal{D}_{\mathbf{P}^n}) \otimes_k \mathcal{D}_X$ where $\pi : \mathbf{P}^n \to \operatorname{Spec} k$. Observe that $\pi_\star(\mathcal{D}_{\mathbf{P}^n}) = R\Gamma(\mathbf{P}^n, \Omega^n_{\mathbf{P}^n}) \simeq k[-n]$ by Serre Duality [Har77, III, Theorem 7.1]. $\square$

4.3. **Good filtrations and singular supports.** Recall that $\mathcal{D}_X$ has an order filtration such that $\mathrm{gr}^F \mathcal{D}_X \simeq \mathrm{Sym}_{\mathcal{O}_X} \Theta_X =: \mathcal{A}$. Let $T^*(X) = \mathbf{Spec}_X \mathcal{A}$ denote the cotangent bundle, i.e., the geometric vector bundle corresponding to sheaf of differentials $\Omega^1_X$.

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. We will consider filtrations $F_0\mathcal{M} \subset F_1\mathcal{M} \subset \cdots \subset \mathcal{M}$ of $\mathcal{O}_X$-submodules such that $\mathcal{M} = \cup F_i\mathcal{M}$ and $(F_i\mathcal{D}_X)(F_j\mathcal{M}) \subset F_{i+j}\mathcal{M}$. Then the associated graded module $\mathrm{gr}^F \mathcal{M}$ is naturally acted on by $\mathcal{A}$. We say that the filtration $F_\bullet\mathcal{M}$ is *good* if $\mathrm{gr}^F \mathcal{M}$ is a coherent $\mathcal{A}$-module. This is equivalent to saying that each $F_i\mathcal{M}$ is a coherent $\mathcal{O}_X$-module and $(F_1\mathcal{D}_X)(F_i\mathcal{M}) = F_{i+1}\mathcal{M}$ for $i \gg 0$. Any $\mathcal{D}_X$-module with a good filtration is coherent. Conversely, if $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$, then it is generated by a coherent $\mathcal{O}_X$-submodule $F_0\mathcal{M}$, and we can define a good filtration by $F_i\mathcal{M} = (F_i\mathcal{D}_X)(F_0\mathcal{M})$.

Using the generic freeness result from commutative algebra and good filtrations, one can deduce the following (cf. [BGK+87, VII, Lemma 9.3], [HTT08, Lemma 3.3.2]).

**Lemma 4.3.1.** *Let $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$. Then there exists an open dense subset $U \subset X$ on which $\mathcal{M}$ is $\mathcal{O}_U$-locally free.*

4.3.2. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ with a good filtration, there is a coherent $\mathcal{O}_{T^*(X)}$-module

$$\widetilde{\operatorname{gr}^F \mathcal{M}} = \mathcal{O}_{T^*(X)} \underset{\pi^{-1}\mathcal{A}}{\otimes} \pi^{-1}(\operatorname{gr}^F \mathcal{M})$$

on $T^*(X) \xrightarrow{\pi} X$ whose direct image is $\operatorname{gr}^F(\mathcal{M})$. We define the singular support $\mathrm{SS}(\mathcal{M})$ to be the support of this module as a closed subscheme of $T^*(X)$ with the *reduced* scheme structure. By standard arguments on filtrations (cf. [HTT08, Theorem 2.2.1, Appendix D]), we have:

**Proposition 4.3.3.** *The singular support does not depend on the choice of a good filtration.*

The singular support is a set-theoretic object; the above proposition is not necessarily true if we did not make $\mathrm{SS}(\mathcal{M})$ reduced (cf. [MS11, Exercise 2.4.2]).

We define the *defect* of $\mathcal{M}$ as $\operatorname{def}(\mathcal{M}) = \dim \mathrm{SS}(\mathcal{M}) - \dim X$.

4.3.4. Take a coherent $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{M}$ generating $\mathcal{M}$ over $\mathcal{D}_X$. We saw in the proof of Proposition 4.1.3 that $\operatorname{supp}(\mathcal{F}) = \operatorname{supp}(\mathcal{M})$. Consider the graded *characteristic ideal* $\mathcal{J} = \operatorname{ann}_{\mathcal{A}}(\operatorname{gr}^F \mathcal{M}) \subset \mathcal{A}$ corresponding to the good filtration $(F_\bullet \mathcal{D}_X)\mathcal{F}$. Since $\operatorname{gr}^F \mathcal{M}$ is generated as an $\mathcal{A}$-module by $\mathcal{F}$, we have $\mathcal{J} \cap \mathcal{O}_X = \operatorname{ann}_{\mathcal{O}_X}(\operatorname{gr}^F \mathcal{M}) = \operatorname{ann}_{\mathcal{O}_X}(\mathcal{F})$. We also deduce from the grading that

$$\mathrm{SS}(\mathcal{M}) \cap T^*_X(X) = \operatorname{supp}(\mathcal{F}) = \operatorname{supp}(\mathcal{M}),$$

where $T^*_X(X) \subset T^*(X)$ is the zero section. Since the closure $\overline{\pi(V(\mathcal{J}))} = V(\mathcal{J} \cap \mathcal{O}_X)$, it now follows that $\pi(\mathrm{SS}(\mathcal{M})) = \operatorname{supp}(\mathcal{M})$ as well.

We also observe from this discussion that $\mathrm{SS}(\mathcal{M})$ is contained in the zero section $T^*_X(X)$ if and only if $\mathcal{M}$ itself is $\mathcal{O}$-coherent.

The singular support $\mathrm{SS}(\mathcal{M})$ is a closed conic inside $T^*(X)$: we have the factorization

$$T^*(X) - T^*_X(X) \to \mathbf{P}(\Omega^1_X) \to X,$$

where the first map is a principal $\mathbf{G}_m$-bundle. Since $\operatorname{gr} \mathcal{M}$ is a graded $\mathcal{A}$-module, it corresponds to a quasi-coherent sheaf $\widetilde{\operatorname{Loc}}(\operatorname{gr} \mathcal{M})$ on $\mathbf{P}(\Omega^1_X)$, which pulls back to $\widetilde{\operatorname{gr} \mathcal{M}}$ on $T^*X - T^*_X(X)$. In this setup, $\mathrm{SS}(\mathcal{M}) - T^*_X(X)$ is the $\mathbf{G}_m$-invariant closed subset projecting to $\operatorname{supp} \widetilde{\operatorname{Loc}}(\operatorname{gr} \mathcal{M})$. If we pick a $k$-point of $X$, then this says that the fiber of $\mathrm{SS}(\mathcal{M})$ (which is now inside $\mathbf{A}^n_k$) is stable under scalar multiplication by $k^\times$. Since $\mathbf{P}(\Omega^1_X)$ is proper over $X$, this shows that $\pi(\mathrm{SS}(\mathcal{M}) - T^*_X(X))$ is closed in $X$.

4.3.5. The functors $\varphi_\star, \varphi^!$ do not preserve $\mathcal{D}$-coherency in general, but there are two cases where we do know the effect of these functors on the singular support.

Let $i : Y \hookrightarrow X$ be a closed embedding. Then a $\mathcal{D}_Y$-module $\mathcal{M}$ is coherent if and only if $i_+(\mathcal{M})$ is coherent as a $\mathcal{D}_X$-module. The forward direction follows from Proposition 4.2.1. Conversely, suppose $i_+(\mathcal{M})$ is coherent. We will work locally, so $i_+(\mathcal{M}) \simeq k[\partial_{r+1}, \dots, \partial_n] \otimes_k i_\bullet(\mathcal{M})$ where $n = \dim X$ and $r = \dim Y$. We can assume our generators are monomials $\{P_j \otimes m_j\}$, and then in turn just take the $\{1 \otimes m_j\}$. The $m_j$ define a map $\mathcal{D}_Y^N \to \mathcal{M}$. Applying $i_+$, we have a map $\mathcal{D}_X^N \to \mathcal{D}_{X\leftarrow Y}^N = i_+(\mathcal{D}_Y^N) \to i_+(\mathcal{M})$. The $j$-th image of $1 \in \mathcal{D}_X$ is $1 \otimes m_j$, so the composed map is just the canonical one and hence surjective. By Kashiwara's theorem, the original map must also be surjective, which shows $\mathcal{M}$ is coherent.

The surjection $i^*\Omega_X \to \Omega_Y$ corresponds to a map $\rho : T^*(X)_Y \to T^*(Y)$ of schemes (vector bundles) over $Y$. Here $T^*(X)_Y = T^*(X) \times_X Y \hookrightarrow T^*(X)$ is a smooth closed subscheme. If $\mathcal{M}$ is coherent, we have

$$\mathrm{SS}(i_+(\mathcal{M})) = \rho^{-1}(\mathrm{SS}(\mathcal{M})) \subset T^*(X)_Y,$$

which can be seen through the characteristic ideals of the filtrations corresponding to the above discussion.

Since $\rho$ is smooth with fibers of dimension $\dim X - \dim Y$, general properties of dimension in flat families [Har77, III, Corollary 9.6] imply that $i_+$ preserves the defect.

4.3.6. Let $\varphi : Y \to X$ be a smooth surjective morphism of schemes. Then a $\mathcal{D}_X$-module $\mathcal{M}$ is coherent if and only if $\varphi^*\mathcal{M}$ is coherent as a $\mathcal{D}_Y$-module. We have a surjection $\mathcal{D}_Y \twoheadrightarrow \mathcal{D}_{Y\to X} : 1 \mapsto 1$, so the forward direction is easy. Conversely, suppose $\varphi^*\mathcal{M} = \mathcal{O}_Y \otimes_{\varphi^{-1}\mathcal{O}_X} \varphi^{-1}\mathcal{M}$ is coherent. By shrinking $X$, we can pick generators of the form $1 \otimes m_j$. So we have a map $\mathcal{D}_X^N \to \mathcal{M}$ such that $\mathcal{D}_Y^N \to \mathcal{D}_{Y\to X}^N = \varphi^*(\mathcal{D}_X^N) \to \varphi^*\mathcal{M}$ is surjective. Faithful flatness of $\varphi$ implies the $m_j$ generate $\mathcal{M}$ as a $\mathcal{D}_Y$-module.

We can apply the exact functor $\varphi^*$ to any good filtration of $\mathcal{M}$ to get a good filtration of $\varphi^*\mathcal{M}$. Now we have an injection $\varphi^*\Omega_X \hookrightarrow \Omega_Y$ which corresponds to a closed embedding $\rho : T^*(X)_Y \hookrightarrow T^*(Y)$. The base change $\varpi : T^*(X)_Y \to T^*(X)$ of $\varphi$ is flat, so applying [GD65, Proposition 2.1.11] to characteristic ideals, we see that

$$\mathrm{SS}(\varphi^*\mathcal{M}) = \rho(\varpi^{-1}(\mathrm{SS}(\mathcal{M})) \subset T^*(Y).$$

We assume $X$ and $Y$ are of pure dimension, so $\varphi$ (and hence $\varpi$) has fibers of dimension $\dim Y - \dim X$ and $\varphi^*$ preserves the defect.

4.3.7. *Bernstein's inequality.* We now come to a remarkable result of Bernstein's that says that the singular support cannot be too "small".

**Theorem 4.3.8** (Bernstein's inequality)**.** *Let $\mathcal{M}$ be a nonzero coherent $\mathcal{D}_X$-module. Then* $\mathrm{def}(\mathcal{M}) \geq 0$*, i.e.,* $\dim \mathrm{SS}(\mathcal{M}) \geq \dim X$*.*

*Proof.* If $\dim \mathrm{SS}(\mathcal{M}) < \dim X$, then $\pi(\mathrm{SS}(\mathcal{M})) = \mathrm{supp}(\mathcal{M})$ is a proper closed subscheme of $X$. By generic smoothness [Har77, III, Corollary 10.7], we can restrict to an open subset to assume $\mathrm{supp}(\mathcal{M})$ is smooth and nonempty. Let $i : Y = \mathrm{supp}(\mathcal{M}) \hookrightarrow X$ be the inclusion. By Kashiwara's theorem and our discussion above, $\mathcal{M} \simeq i_+(\mathcal{N})$ for a coherent $\mathcal{D}_Y$-module $\mathcal{N}$ with $\mathrm{def}(\mathcal{N}) = \mathrm{def}(\mathcal{M}) < 0$. We get a contradiction by induction on $\dim X$. □

4.4. **Functors for holonomic $\mathcal{D}$-modules.** We define a coherent $\mathcal{D}_X$-module $\mathcal{M}$ to be *holonomic* if $\mathrm{def}(\mathcal{M}) \leq 0$. By Bernstein's inequality, this says that $\mathcal{M}$ is either zero or $\mathrm{def}(\mathcal{M}) = 0$, i.e., $\mathcal{M}$ has "minimal possible size". Let $\mathrm{Mod}_h(\mathcal{D}_X) \subset \mathrm{Mod}_c(\mathcal{D}_X)$ denote the full subcategory of holonomic $\mathcal{D}_X$-modules. For example, any $\mathcal{O}$-coherent $\mathcal{D}_X$-module is holonomic.

**Proposition 4.4.1.**

(i) *The subcategory $\mathrm{Mod}_h(\mathcal{D}_X)$ is abelian.*
(ii) *Any holonomic $\mathcal{D}_X$-module has finite length.*
(iii) *If $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$, then there exists an open dense subset $U \subset X$ such that $\mathcal{F}|_U$ is an $\mathcal{O}$-coherent $\mathcal{D}_U$-module.*

*Proof.* Cf. [HTT08, Propositions 3.1.2, 3.1.6]. We remark that in (iii), $U$ may be taken to be the complement of the closed subset $\pi(\mathrm{SS}(\mathcal{M}) - T^*_X(X)) \subset X$. □

4.4.2. We know that $\mathcal{D}$-modules satisfy the property of smooth descent (see 2.3.2). It follows from 4.3.6 that coherent and holonomic $\mathcal{D}$-modules also have this property.

4.4.3. We say that a complex $\mathcal{M}^\bullet \in D^b_c(\mathcal{D}_X)$ is holonomic if all of its cohomology sheaves are holonomic $\mathcal{D}_X$-modules. Let the full triangulated subcategory consisting of these complexes be denoted $D^b_h(\mathcal{D}_X) \subset D^b_c(\mathcal{D}_X)$.

Beilinson [Beĭ87] has shown that the natural morphism $D^b(\mathrm{Mod}_h(\mathcal{D}_X)) \to D^b_h(\mathcal{D}_X)$ is an equivalence, but we will not need this.

We will spend most of the rest of this section giving an overview of the proof of the following main theorem.

**Theorem 4.4.4.** *Let $\varphi : Y \to X$ be a morphism of schemes. Then*

$$\varphi_\star : D_h^b(\mathcal{D}_Y) \to D_h^b(\mathcal{D}_X) \qquad \textit{and} \qquad \varphi^! : D_h^b(\mathcal{D}_X) \to D_h^b(\mathcal{D}_Y).$$

The proof is based on the following special case:

**Lemma 4.4.5.** *Let $i : Y \hookrightarrow X$ be a locally closed embedding. Then $i_\star D_h^b(\mathcal{D}_Y) \subset D_h^b(\mathcal{D}_X)$.*

*Proof of Theorem 4.4.4 for $\varphi^!$.* It suffices to prove the theorem for $\varphi$ a smooth morphism and a closed embedding. Exactness of $\varphi^*$ and 4.3.6 imply the smooth case.

So consider $Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$ where $i$ is a closed embedding and $U$ is the complement. Then for $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_Y)$, we have a distinguished triangle $i_\star i^!(\mathcal{M}^\bullet) \to \mathcal{M}^\bullet \to j_\star(\mathcal{M}^\bullet|_U)$ from (3.4.1.1) and Corollary 3.4.9. By the Key Lemma 4.4.5, we have that $j_\star(\mathcal{M}^\bullet|_U)$ is a holonomic complex. Therefore $i_\star i^!(\mathcal{M}^\bullet) \in D_h^b(\mathcal{D}_X)$. Exactness of $i_+$ and 4.3.5 imply that $i^!(\mathcal{M}^\bullet)$ is holonomic. □

4.4.6. **Criterion of holonomicity.** *Let $\mathcal{M}^\bullet \in D_{qc}^b(\mathcal{D}_X)$. Then $\mathcal{M}^\bullet$ is holonomic if and only if $\mathcal{M}^\bullet$ is coherent and there exists a dense subset $V \subset X$ (e.g., $V = X(k)$) such that $i_x^!(\mathcal{M}^\bullet)$ has finite dimensional cohomology for all $x \in V$.*

The criterion says that holonomicity can be checked fiberwise. For the proof, see [Ber84, 3.3]. Recall the notion of a smooth stratification introduced in 3.5.5. Using the same notation, the proof of the criterion also gives the following.

**Corollary 4.4.7.** *A complex $\mathcal{M}^\bullet \in D_{qc}^b(\mathcal{D}_X)$ is holonomic if and only if there exists a smooth stratification $X = Z_0 \supset Z_1 \supset \cdots$ such that all $i_\ell^!(\mathcal{M}^\bullet)$ have $\mathcal{O}$-coherent cohomology.*

*Proof of Theorem 4.4.4 for $\varphi_\star$.* The Key Lemma contains the case when $\varphi$ is a closed embedding, so we will only consider when $\varphi : Y = X \times T \to X$ is a proper projection. For $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_Y)$, proper pushforward (Proposition 4.2.1) implies that $\mathcal{N}^\bullet := \varphi_\star(\mathcal{M}^\bullet)$ is coherent. For a closed point $x \in X(k)$, denote $i_{T_x} : T_x = \{x\} \times T \hookrightarrow Y$ the closed embedding and $\pi_x : T_x \to \{x\}$ the projection to the point. By base change (Theorem 3.7.1), $i_x^!(\mathcal{N}^\bullet) \simeq \pi_{x\star} i_{T_x}^!(\mathcal{M}^\bullet)$. We know that $i_{T_x}^!$ preserves holonomicity, so $i_{T_x}^!(\mathcal{M}^\bullet)$ is in particular coherent. Using proper pushforward again, we get that $\pi_{x\star} i_{T_x}^!(\mathcal{M}^\bullet)$ has finite dimensional cohomology. We conclude that $\mathcal{N}^\bullet$ is holonomic using the fiber criterion. □

4.4.8. *Duality functor.* Over an arbitrary sheaf of rings $\mathcal{R}$ on $X$, there is a possible ambiguity in what "locally projective" means (for some cover vs. for any cover). In general, it seems $R\mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \bullet)$ behaves best when $\mathcal{F}$ is locally finitely presented (cf. [Lur, §8]). So we will only consider $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \bullet)$ for $\mathcal{M}^\bullet \in D_c^b(\mathcal{D}_X)$. Then $\mathcal{M}^\bullet$ is isomorphic to some bounded complex $\mathcal{P}^\bullet$ of coherent locally projective $\mathcal{D}_X$-modules. In this "nice" case, $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}^\bullet, \bullet)$ does preserves acyclics, so by general derived category theory $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \bullet) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}^\bullet, \bullet)$.

Define the *duality functor* $\mathbb{D} : D_c^b(\mathcal{D}_X)^{\mathrm{op}} \to D_c^b(\mathcal{D}_X)$ by

$$\mathbb{D}(\mathcal{M}^\bullet) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X^\Omega)[\dim X].$$

For a coherent locally projective $\mathcal{D}_X$-module $\mathcal{P}$, we have that $\mathcal{P}^\vee := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{D}_X^\Omega)$ is a left $\mathcal{D}_X$-module via the second action on $\mathcal{D}_X^\Omega$. Now $\mathcal{P}$ is locally a direct summand of a free $\mathcal{D}_X$-module of finite rank, from which we deduce that $\mathcal{P}^\vee \simeq \overleftarrow{\Omega}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{D}_X)$ is coherent and locally projective, and $(\mathcal{P}^\vee)^\vee \xleftarrow{\sim} \mathcal{P}$. Therefore $\mathbb{D}^2 \simeq \mathrm{id}$. By definition, $H^i\mathbb{D}(\mathcal{M}) = \mathcal{E}xt_{\mathcal{D}_X}^{\dim X + i}(\mathcal{M}, \mathcal{D}_X^\Omega)$ for $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$.

The following theorem of Roos allows us to show that $\mathbb{D}$ is well-behaved.

**Theorem 4.4.9** (Roos)**.** *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then*

(i) $\operatorname{codim}_{T^*(X)} \mathrm{SS}(\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega)) \geq i$ *for all* $i$,
(ii) $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega) = 0$ *for* $i < \operatorname{codim} \mathrm{SS}(\mathcal{M})$.

*Proof.* Cf. [HTT08, Theorem 2.6.7]. We may assume $X$ is affine. The theorem is reminiscent of the analogous result in commutative algebra for regular rings. Indeed, what the proof of Proposition 3.2.2 really showed was that we can find a good filtration of $\mathcal{M}$ such that $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is a subquotient of $\mathcal{E}xt^i_{\mathcal{A}}(\operatorname{gr}^F \mathcal{M}, \mathcal{A})$. This reduces the claim to the commutative setting. □

**Corollary 4.4.10.** *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module.*
(i) $\mathbb{D}(\mathcal{M})$ *lives cohomologically in degrees* $[-\dim X, 0]$.
(ii) $\mathcal{M}$ *has a coherent locally projective resolution of length* $\leq \dim X$.
(iii) $\mathcal{M}$ *is holonomic if and only if* $H^i\mathbb{D}(\mathcal{M}) = 0$ *for* $i \neq 0$.
(iv) $H^0\mathbb{D}$ *gives an auto-duality* $\mathrm{Mod}_h(\mathcal{D}_X)^{\mathrm{op}} \to \mathrm{Mod}_h(\mathcal{D}_X)$.

*Proof.* Cf. [Ber84, 3.6]. We expound on point (ii). It suffices to show that when $X$ is affine, $\mathcal{M}$ has projective dimension $\leq \dim X$. Let $\mathcal{P}^\bullet \to \mathcal{M}$ be a bounded coherent projective resolution (living in degrees $\leq 0$). Then $\mathbb{D}\mathcal{P}^\bullet$ consists of coherent projective modules, lives in degrees $\geq -\dim X$, and is acyclic in degrees $> 0$. Projectivity implies that $\mathbb{D}\mathcal{P}^\bullet \simeq \tau^{\leq 0}(\mathbb{D}\mathcal{P}^\bullet) \oplus \tau^{>0}(\mathbb{D}\mathcal{P}^\bullet)$ as complexes, where each summand still consists of coherent projectives. Then $\mathbb{D}\tau^{>0}\mathbb{D}\mathcal{P}^\bullet = 0$, so we have $\mathcal{P}^\bullet \simeq \mathbb{D}^2\mathcal{P}^\bullet \simeq \mathbb{D}\tau^{\leq 0}\mathbb{D}\mathcal{P}^\bullet$, where the last complex consists of coherent projectives living in degrees $[-\dim X, 0]$. But then since $K^{b,proj}(\mathcal{D}_X) \simeq D^b(\mathcal{D}_X)$, we have a map of complexes $\mathbb{D}\tau^{\leq 0}\mathbb{D}\mathcal{P}^\bullet \to \mathcal{P}^\bullet \to \mathcal{M}$, giving the desired resolution. □

Note that (iv) implies that $\mathbb{D}$ preserves $D^b_h(\mathcal{D}_X)$.

4.5. **Lemma on $b$-functions.** We now prove the Key Lemma 4.4.5. The case of a closed embedding follows from 4.3.5. Hence we assume that $j : U \hookrightarrow X$ is an open embedding. We may further assume that $X$ is affine. It suffices to prove that $j_\star(\mathcal{M})$ is a holonomic $\mathcal{D}_X$-complex for a single object $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_U)$. Cover $U$ by open affine subsets $U_{f_\alpha}$ for $f_\alpha \in \Gamma(X, \mathcal{O}_X)$. We can replace $\mathcal{M}$ by its Čech resolution, which consists of objects $(j_{\alpha_1,\ldots,\alpha_i})_\star(\mathcal{M}|_{U_{f_{\alpha_1}\cdots f_{\alpha_i}}})$, where $j_{\alpha_1,\ldots,\alpha_i} : U_{f_{\alpha_1}\cdots f_{\alpha_i}} \hookrightarrow U$ is an affine morphism. Thus by composition we reduce to the case where $U = U_f$ is affine for some $f \in \Gamma(X, \mathcal{O}_X)$. In this case $j_\star = j_\bullet$ is exact. Finite generation allows us to assume that $\mathcal{M}$ is generated by a single section $m_0 \in \Gamma(U, \mathcal{M})$.

4.5.1. The hard (and surprising) part of the lemma is that $j_\star(\mathcal{M})$ is coherent as a $\mathcal{D}_X$-module. This is far from true for $\mathcal{O}$-modules. By quasi-coherence of $\mathcal{D}_X$, we see that $\mathcal{D}(U) = \cup_{n=-\infty}^0 \mathcal{D}(X) f^n$. Therefore $j_\star(\mathcal{M})$ is generated as a $\mathcal{D}_X$-module by the sections $f^n m_0$ for all $n \in \mathbf{Z}$. We now come to the following remarkable result:

**Lemma 4.5.2** (Lemma on $b$-functions)**.** *Suppose we have* $X$ *affine,* $U = U_f \xhookrightarrow{j} X$ *for some* $f \in \Gamma(X, \mathcal{O}_X)$*, a holonomic* $\mathcal{D}_U$*-module* $\mathcal{M}$*, and a section* $m_0 \in \Gamma(U, \mathcal{M}) = \Gamma(X, j_\star(\mathcal{M}))$*. Then there exists* $d_0 \in \mathcal{D}(X)[s]$ *and a nonzero polynomial* $b_0 \in k[s]$ *such that for all* $n \in \mathbf{Z}$ *we have*
$$d_0(n)(f^{n+1}m_0) = b_0(n) f^n m_0.$$

Note that if we replaced $\mathcal{M}$ by $\mathcal{O}_U$, then the statement of the lemma does not involve holonomicity, yet the result is still nontrivial. The proof, however, relies heavily on the theory of holonomic $\mathcal{D}$-modules. In particular, it uses the following analogue of Lemma 4.1.1.

**Lemma 4.5.3.** *If* $\mathcal{M}$ *is a* $\mathcal{D}_X$*-module,* $U \subset X$ *an open subset, and* $\mathcal{N}_U \subset \mathcal{M}|_U$ *a holonomic* $\mathcal{D}_U$*-modules, then there exists a holonomic* $\mathcal{D}_X$*-submodule* $\mathcal{N} \subset \mathcal{M}$ *such that* $\mathcal{N}|_U = \mathcal{N}_U$.

*Proof.* Cf. [Ber84, 3.7] and [HTT08, Proposition 3.1.7]. □

We refer the reader to [Lic09, Theorem 2.1] for the proof of the lemma on $b$-functions. We found the notation and reformulation used there, which follows [BG08], to be the most intuitive. Another good proof is given in [Gin98, Lemma 4.2.2].

*Remark* 4.5.4. The generator of the principal ideal of all polynomials $b_0 \in k[s]$ satisfying the formula in the lemma is called the *$b$-function*. By using Hironaka's theorem on resolution of singularities, Kashiwara [Kas03, Theorem 6.9] showed that the $b$-function has negative rational zeros.

4.5.5. We now finish the proof of the Key Lemma 4.4.5. To do so, we will first summarize the necessary notation and results from the proof of the lemma on $b$-functions. Define the special $\mathcal{D}_U[s]$-module "$f^s$" to be the free $\mathcal{O}_U[s]$-module of rank 1, generated by a formal symbol $f^s$. We give it the $\mathcal{D}_U$-action induced by the formula

$$\xi(f^s) = s\xi(f)f^{-1} \cdot f^s,$$

for any $\xi \in \Theta_U$. Consider the $\mathcal{D}_U[s]$-module $\mathcal{M} \otimes \text{“}f^s\text{”} := \mathcal{M} \otimes_{\mathcal{O}_U} \text{“}f^s\text{”}$ and the $\mathcal{D}_X[s]$-module

$$j_\star(\mathcal{M} \otimes \text{“}f^s\text{”}).$$

There is a morphism of $\mathcal{D}_X$-modules $\mathrm{ev}_{s=n} : j_\star(\mathcal{M} \otimes \text{“}f^s\text{”}) \to j_\star(\mathcal{M})$ for any integer $n$, which sends $m \otimes g(s) \cdot f^s \mapsto g(n)f^n m$.

Let $K = k(s)$ be the field of rational functions and $\widehat{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ the base change. The lemma on $b$-functions is proved by showing that $j_\star(\mathcal{M} \otimes \text{“}f^s\text{”}) \otimes_{k[s]} K$, considered as a $\mathcal{D}_X(s) \simeq \mathcal{D}_{\widehat{X}}$-module over $K$, is generated by $m_0 \otimes f^s$ and holonomic[5].

Let $\mathcal{J} \subset \mathcal{A}[s]$ be an ideal such that $\mathcal{J} \otimes_{k[s]} K = \operatorname{ann}_{\mathcal{A}(s)} \operatorname{gr}^F(j_\star(\mathcal{M} \otimes \text{“}f^s\text{”}) \otimes_{k[s]} K)$ with respect to the good filtration induced by the generator $m_0 \otimes f^s$. So

$$\mathbf{Spec}_X(\mathcal{A}[s]/\mathcal{J}) \to \operatorname{Spec} k[s]$$

is a morphism of schemes whose generic fiber is (set-theoretically) the singular support of $j_\star(\mathcal{M} \otimes \text{“}f^s\text{”}) \otimes_{k[s]} K$. The dimension of this fiber is $\leq \dim \widehat{X} = \dim X$ by holonomicity. By upper semi-continuity of fiber dimension, there are only finitely many points of $\operatorname{Spec} k[s]$ whose fiber has dimension higher than $\dim X$. So if we just choose any integer $n$ outside of those points, we get $\dim \mathbf{Spec}_X(\mathcal{A}/\operatorname{ev}_{s=n}(\mathcal{J})) \leq \dim X$. Since $\operatorname{ev}_{s=n}(\mathcal{J}) \subset \operatorname{ann}_{\mathcal{A}}(f^n m_0)$ with respect to the filtration $(F_\bullet \mathcal{D}_X) f^n m_0$, we deduce that $\mathcal{D}_X f^n m_0 \subset j_\star(\mathcal{M})$ is holonomic. Since any $f^n m_0$ generates $j_\star(\mathcal{M})$ for $n \ll 0$ by the lemma on $b$-functions, we conclude that $j_\star(\mathcal{M})$ is holonomic.

4.6. **Duality.** For a morphism $\varphi : Y \to X$ of schemes, we define two new functors

$$\varphi_! = \mathbb{D}\varphi_\star\mathbb{D} : D^b_h(\mathcal{D}_Y) \to D^b_h(\mathcal{D}_X) \qquad \text{and} \qquad \varphi^\star = \mathbb{D}\varphi^!\mathbb{D} : D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_Y).$$

To summarize, we have six functors $\varphi^!, \varphi^\star, \varphi_!, \varphi_\star, \mathbb{D}_X, \mathbb{D}_Y$ acting on the derived categories of bounded holonomic complexes. They are related in a manner entirely analogous to Verdier duality for sheaves and Grothendieck's coherent duality:

**Theorem 4.6.1.** *The functors on $D^b_h(\mathcal{D}_Y)$ and $D^b_h(\mathcal{D}_X)$ are related as follows:*

(i) *There exists a canonical morphism of functors $\varphi_! \to \varphi_\star$ which is an isomorphism if $\varphi$ is proper.*
(ii) *The functor $\varphi_!$ is left adjoint to $\varphi^!$.*
(iii) *The functor $\varphi^\star$ is left adjoint to $\varphi_\star$.*
(iv) *If $\varphi$ is smooth, then $\varphi^! = \varphi^\star[2(\dim Y - \dim X)]$.*

[5]This discussion does not require $K$ to be algebraically closed, but one could replace $K$ with $\overline{K}$ if desired.

Note that if $\varphi$ is proper (resp. smooth), then $\varphi_!$ (resp. $\varphi^\star$) extends to coherent complexes. Theorem 4.6.1 follows from more general statements in these contexts.

**Theorem 4.6.2.** *Let $\varphi : Y \to X$ be a proper morphism. Then we have a canonical isomorphism of functors $\varphi_\star \mathbb{D}_Y \simeq \mathbb{D}_X \varphi_\star : D^b_c(\mathcal{D}_Y) \to D^b_c(\mathcal{D}_X)$. The functor $\varphi_\star$ is left adjoint to $\varphi^!$.*

*Proof.* Cf. [Ber84, 3.10], [HTT08, Theorem 2.7.2, Corollary 2.7.3], and [BGK$^+$87, Theorem 9.12]. The morphism of functors $\varphi_\star \mathbb{D} \to \mathbb{D}\varphi_\star$ is constructed by considering the cases of $\varphi$ a closed embedding and a proper projection separately. One must then show that the resulting morphism is independent of the choice of decomposition. The proof of this is sketched in [BGK$^+$87, 9.6.2]. □

**Theorem 4.6.3.** *Let $\varphi : Y \to X$ be a smooth morphism. There exists a canonical isomorphism of functors $\mathbb{D}_Y \varphi^! \simeq \varphi^! \mathbb{D}_X[-2(\dim Y - \dim X)] : D^b_c(\mathcal{D}_X) \to D^b_c(\mathcal{D}_Y)$.*

*Proof.* Cf. [Ber84, 3.13], [HTT08, Theorem 2.7.1], and [BGK$^+$87, Proposition 9.13]. □

*Proof of Theorem 4.6.1.* Cf. [Ber84, 3.9], [HTT08, Theorems 3.2.14, 3.2.16], and [BGK$^+$87, Theorem 10.2]. Most of the work has already been done in the previous two theorems. □

## 5. Group actions

Let $X$ be a scheme endowed with a left action $\mathrm{act} : G \times X \to X$ of an affine group scheme of finite type over $k$. Denote the group multiplication by $\mathrm{mult} : G \times G \to G$. The quotient stack $G\backslash X$ is an algebraic $k$-stack with schematic diagonal [Wan11, Theorem 2.0.2]. We will introduce $G$-equivariant objects by viewing them as corresponding objects on $G\backslash X$.

5.1. **Equivariant quasi-coherent sheaves.** To warm up, let us review the theory of $G$-equivariant quasi-coherent $\mathcal{O}_X$-modules through the stack perspective.

5.1.1. For a scheme $Y$, let $\mathrm{QCoh}(Y)$ denote the category of quasi-coherent $\mathcal{O}_Y$-modules. Then QCoh forms a stack over the big fpqc site $(\mathbf{Sch}_{/k})_{\mathrm{fpqc}}$ by [FGI$^+$05, Theorem 4.23]. Let $\mathcal{X}$ be an algebraic $k$-stack with schematic diagonal. We define a quasi-coherent sheaf on $\mathcal{X}$ to be a 1-morphism of stacks $\mathcal{X} \xrightarrow{\mathcal{F}} \mathrm{QCoh}$, i.e., specifying for every 1-morphism $S \xrightarrow{\mathcal{P}} \mathcal{X}$ a quasi-coherent sheaf $\mathcal{F}_{\mathcal{P}}$ on $S$ in a way compatible with 2-morphisms and composition. Fix an fpqc 1-morphism $X \to \mathcal{X}$. Then flat descent says the above definition is equivalent to giving a sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ together with an isomorphism $\tau_{\mathcal{F}} : \mathrm{pr}_1^* \mathcal{F} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{F}$ satisfying the cocycle condition.

Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Then considering a quasi-coherent sheaf on $\mathcal{X}$ as a 1-morphism $\mathcal{X} \xrightarrow{\mathcal{F}} \mathrm{QCoh}$, we define the pullback $\varphi^* \mathcal{F}$ simply as the composition $\mathcal{Y} \to \mathcal{X} \to \mathrm{QCoh}$. This is compatible with the usual pullback on schemes.

5.1.2. Now we consider when our stack is the quotient stack $G\backslash X$. Then the trivial bundle together with the action map $G \times X \to X$ defines an fpqc 1-morphism $X \to G\backslash X$. We have an isomorphism $(\mathrm{act}, \mathrm{pr}_2) : G \times X \to X \times_{G\backslash X} X$ where the corresponding 2-morphism is the map $G \times G \times X \to G \times G \times X : (g_1, g_2, x) \mapsto (g_1 g_2, g_2, x)$ of trivial $G$-bundles over $G \times X$. Using this isomorphism, we can identify

$$G \times (G \times X) \xrightarrow[\sim]{\mathrm{id} \times (\mathrm{act}, \mathrm{pr}_2)} G \times (X \underset{G\backslash X}{\times} X) \simeq (G \times X) \underset{\mathrm{pr}_2, G\backslash X}{\times} X \xrightarrow[\sim]{(\mathrm{act}, \mathrm{pr}_2) \times \mathrm{id}} X \underset{G\backslash X}{\times} X \underset{G\backslash X}{\times} X.$$

Under these identifications, $\mathrm{pr}_{13}, \mathrm{pr}_{23}, \mathrm{pr}_{12}$ correspond to the morphisms $\mathrm{mult} \times \mathrm{id}, \mathrm{pr}_{23}, \mathrm{id} \times \mathrm{act}$ respectively on $G \times G \times X \to G \times X$. Therefore we see that a quasi-coherent sheaf on $G\backslash X$ is

the data of $\mathcal{F} \in \mathrm{QCoh}(X)$ and an isomorphism $\tau_{\mathcal{F}} : \mathrm{act}^*\mathcal{F} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{F}$ such that the diagram
(5.1.2.1)

$$\begin{tikzcd}
(\mathrm{id}\times\mathrm{act})^*\mathrm{act}^*(\mathcal{F}) \arrow[r, "(\mathrm{id}\times\mathrm{act})^*(\tau_{\mathcal{F}})"] \arrow[d, "\sim"'] & (\mathrm{id}\times\mathrm{act})^*\mathrm{pr}_2^*(\mathcal{F}) \arrow[r, "\sim"] & \mathrm{pr}_{23}^*\mathrm{act}^*(\mathcal{F}) \arrow[d, "\mathrm{pr}_{23}^*(\tau_{\mathcal{F}})"] \\
(\mathrm{mult}\times\mathrm{id})^*\mathrm{act}^*(\mathcal{F}) \arrow[r, "(\mathrm{mult}\times\mathrm{id})^*(\tau_{\mathcal{F}})"] & (\mathrm{mult}\times\mathrm{id})^*\mathrm{pr}_2^*(\mathcal{F}) \arrow[r, "\sim"] & \mathrm{pr}_{23}^*\mathrm{pr}_2^*(\mathcal{F})
\end{tikzcd}$$

of sheaves on $G \times G \times X$ commutes. A morphism of sheaves on $X\backslash G$ is a morphism between $\mathcal{F}$'s compatible with the $\tau$'s.

5.1.3. Fix a scheme $S$. For an $S$-point $x \in X(S)$, i.e., a morphism of schemes $S \xrightarrow{x} X$, let $\mathcal{F}_x := x^*(\mathcal{F})$, which we think of as the $S$-point "fiber". Then for $g \in G(S)$ and $x \in X(S)$, we have an isomorphism $\tau_{g,x} := (g,x)^*(\tau_{\mathcal{F}})$ between $\mathcal{F}_{gx} \xrightarrow{\sim} \mathcal{F}_x$. The cocycle condition requires that for $g_1, g_2 \in G(S)$ and $x \in X(S)$, the composition

$$\mathcal{F}_{g_1g_2x} \xrightarrow{\tau_{g_1,g_2x}} \mathcal{F}_{g_2x} \xrightarrow{\tau_{g_2,x}} \mathcal{F}_x$$

must equal $\tau_{g_1g_2,x}$. Conversely, taking $S = G \times X$ and $(g,x) = \mathrm{id}_{G\times X}$ shows that $\tau_{g,x} = \tau_{\mathcal{F}}$, so we have given another equivalent description of a quasi-coherent sheaf on $G\backslash X$. This description is more geometrically intuitive, as one can see that what we really have is a $G$-action on the fibers of $\mathcal{F}$ compatible with the action on $X$.

Note that since $\tau_{g,x}$ is an isomorphism, we deduce from associativity that $\tau_{1,x} = \mathrm{id}$. From this we deduce that the restriction of $\tau_{\mathcal{F}}$ to $\{1\} \times X \subset G \times X$ is the identity map $\mathrm{id}_{\mathcal{F}}$. Since $G(S)$ has inverses, we can also deduce from the fiber description that if we started with a morphism $\tau_{\mathcal{F}} : \mathrm{act}^*\mathcal{F} \to \mathrm{pr}_2^*\mathcal{F}$, not *a priori* assumed to be an isomorphism, such that the restriction to $\{1\} \times X$ is identity and the associativity condition of (5.1.2.1) holds, then $\tau_{\mathcal{F}}$ must be an isomorphism.

**Definition 5.1.4.** A *$G$-equivariant quasi-coherent sheaf* is a sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ together with a morphism $\tau_{\mathcal{F}} : \mathrm{act}^*\mathcal{F} \to \mathrm{pr}_2^*\mathcal{F}$ satisfying any of the equivalent conditions mentioned above. We denote the category of $G$-equivariant sheaves by $\mathrm{QCoh}(X, G)$.

So by flat descent (or one could just take this as the definition), we have an equivalence of categories between $\mathrm{QCoh}(X, G)$ and the category $\mathrm{QCoh}(G\backslash X)$ of quasi-coherent sheaves on the quotient stack $G\backslash X$.

5.1.5. Let us try to provide some more intuition. Suppose for the moment that $X = \mathrm{Spec}\, A$ is affine. Then act corresponds to a map of rings $\mathrm{act}^* : A \to \mathcal{O}(G) \otimes A$. Let $M$ be a $G$-equivariant $A$-module. Then $\tau_M$ is a map of $\mathcal{O}(G) \otimes A$-modules

$$(\mathcal{O}(G) \otimes A) \underset{\mathrm{act}^*, A}{\otimes} M \to \mathcal{O}(G) \underset{k}{\otimes} M.$$

By tensor adjunction, this is equivalent to an $A$-linear morphism $\Delta_M : M \to \mathcal{O}(G) \otimes_k M$. Note, however, that $A$ acts on the right hand side by $\mathrm{act}^*$, i.e., for $f \in A$ and $m \in M$, we have $\Delta_M(fm) = \mathrm{act}^*(f)\Delta_M(m)$. The cocycle condition on $\tau_M$ is then equivalent to requiring that

$$(\mathrm{id}_{\mathcal{O}(G)} \otimes \Delta_M)\Delta_M = (\mathrm{mult}^* \otimes \mathrm{id}_M)\Delta_M \qquad \text{and} \qquad (\mathrm{ev}_1 \overline{\otimes}\, \mathrm{id}_M)\Delta_M = \mathrm{id}_M\, .$$

This makes $M$ a (left) comodule (cf. [Jan03, I, 2.8]) over the Hopf algebra $\mathcal{O}(G)$. Therefore we see that $G$-equivariance of $M$ is equivalent to giving it the structure of a $G$-module (=representation) such that the $G$-action on $M$ is "compatible" with the $A$-action through $\mathrm{act}^*$. For $g \in G$, the (left) action on $M$ is given by $g \cdot m = \tau_{g^{-1}}(m)$

5.1.6. We return to the case of a general scheme $X$ and $\mathcal{F} \in \mathrm{QCoh}(X,G)$. Let $U \overset{j}{\hookrightarrow} X$ be an open affine subset. The problem is that $U$ does not have to be preserved by $G$, so we cannot directly apply our observations above. However, we can still see what happens once we pick a particular element $g \in G(k)$ to act with (here we could extend to an arbitrary base by base change; we chose $g$ to be a geometric point for simplicity and intuition). Then the action by $g$ sends $\mathrm{act}_g : U \xrightarrow{\sim} gU$, and $\tau_{g,j}$ gives an isomorphism of $\mathcal{O}_U$-modules $\mathrm{act}_g^*(\mathcal{F}|_{gU}) \simeq \mathcal{F}|_U$. If we take local sections $\Gamma(U, \bullet)$, this becomes a map of $\mathcal{O}(U)$-modules

$$\mathcal{O}(U) \underset{\mathcal{O}(gU)}{\otimes} \Gamma(gU, \mathcal{F}) \xrightarrow{\sim} \Gamma(U, \mathcal{F}).$$

Another way of saying this is that we have an $\mathrm{act}_g^*$-isomorphism $\Gamma(gU, \mathcal{F}) \to \Gamma(U, \mathcal{F})$.

5.1.7. *Representations on global sections.* Assume $X$ is quasi-compact and quasi-separated. By gluing maps together from the local situation, we can always give the space of *global* sections $\Gamma(X, \mathcal{F})$ a $G$-module structure. Explicitly, applying $\Gamma(G \times X, \bullet)$ to $\tau_{\mathcal{F}}$, we get the following morphism

$$\Delta_{\mathcal{F}} : \Gamma(X, \mathcal{F}) \overset{1 \otimes \bullet}{\to} \Gamma(G \times X, \mathrm{act}^* \mathcal{F}) \overset{\Gamma(\tau_{\mathcal{F}})}{\to} \Gamma(G \times X, \mathrm{pr}_2^* \mathcal{F}) \simeq \mathcal{O}(G) \underset{k}{\otimes} \Gamma(X, \mathcal{F})$$

where the last step uses the projection formula and that global sections commutes with infinite direct sums since $X$ is quasi-compact and quasi-separated. The cocycle condition on $\tau_{\mathcal{F}}$ ensures that $\Gamma(X, \mathcal{F})$ becomes a $G$-module.

It is easier to see what $\Delta_{\mathcal{F}}$ does if we look at the local picture. Let $U \subset X$ be an open affine subset. Then $\mathrm{act}(G \times U) = G \cdot U$ is open since $G/k$ is fppf, and $\Delta_{\mathcal{F}}$ sends

$$\Gamma(G \cdot U, \mathcal{F}) \to \Gamma(G \times U, \mathcal{O}_{G \times X}) \underset{\mathrm{act}^*, \Gamma(G \cdot U, \mathcal{O}_X)}{\otimes} \Gamma(G \cdot U, \mathcal{F}) \to \mathcal{O}(G) \underset{k}{\otimes} \Gamma(U, \mathcal{F}).$$

Since $\tau_{\mathcal{F}}$ is compatible with taking stalks, if we evaluate (i.e., take the fiber) at $g \in G(k)$, then we are restricting our global section to $\Gamma(gU, \mathcal{F})$ and then applying $\tau_{\mathcal{F}}$ to get a local section in $\Gamma(U, \mathcal{F})$, as described earlier. These local sections then patch across different $U$ to give a global section.

Note that the $G$-action is not $\mathcal{O}(X)$-linear: for a fixed $g$, the action of $g^{-1}$ is an $\mathrm{act}_g^*$-morphism. In other words, the $G$-module structure is compatible with the $\mathcal{O}_X$-module structure on $\mathcal{F}$ via act, as we already noted for the affine case.

**Example 5.1.8.** The structure sheaf $\mathcal{O}_X$ has a $G$-equivariant structure via the canonical isomorphisms $\mathrm{act}^* \mathcal{O}_X \simeq \mathcal{O}_{G \times X} \simeq \mathrm{pr}_2^* \mathcal{O}_X$. Note that in this case $\mathrm{act}^* : \mathrm{act}^{-1} \mathcal{O}_X \to \mathcal{O}_{G \times X}$ is the morphism of structure rings attached to the morphism of ringed spaces $\mathrm{act} : G \times X \to X$. Thus the $G$-action on global sections is the natural one: for a regular function $f \in \Gamma(X, \mathcal{O}_X)$ and a geometric point $g \in G(k)$, the regular function $g \cdot f \in \Gamma(X, \mathcal{O}_X)$ sends $x \mapsto f(g^{-1}x)$ for $x \in X(k)$.

5.1.9. **Infinitesimal actions.** The ideas here are mainly from [Kem78, §1]. Just as we consider the Lie algebra representation corresponding to a group representation, we would like to attach some kind of Lie algebra action to a $G$-equivariant sheaf.

Let $\mathfrak{m} \subset \mathcal{O}(G)$ denote the maximal ideal corresponding to the identity $1 \in G(k)$. Let $G_i = \mathrm{Spec}(\mathcal{O}(G)/\mathfrak{m}^{i+1})$ be the $i$-th infinitesimal neighborhood of 1, which is a closed subscheme (but not a subgroup) of $G$. Then act restricts to a map $\mathrm{act}_i : G_i \times X \to X$ which is the identity on topological spaces. In particular for an open subset $U \subset X$ we always have $\mathrm{act}_i^{-1}(U) = G_i \times U$, which is homeomorphic to $U$. Therefore we have no problems applying our discussion of 5.1.5 to get morphisms

$$\Delta_{i,U} : \Gamma(U, \mathcal{F}) \to \Gamma(G_i \times U, \mathrm{act}_i^* \mathcal{F}) \overset{\Gamma(\tau_{\mathcal{F}})}{\to} \Gamma(G_i \times U, \mathrm{pr}_2^* \mathcal{F}) \simeq \mathcal{O}(G_i) \underset{k}{\otimes} \Gamma(U, \mathcal{F}).$$

We can do this for all open subsets $U$, and the maps are clearly compatible. Thus we get a map of sheaves $\Delta_i : \mathcal{F} \to \mathcal{O}(G_i) \otimes_k \mathcal{F}$. The $\Delta_i$ are compatible with the inclusions $G_i \hookrightarrow G_j$ for $i \leq j$, so the inverse limit $\mathcal{F} \to \widehat{\mathcal{O}(G)} \otimes_k \mathcal{F}$ makes $\mathcal{F}$ into a comodule over the formal group $\widehat{\mathcal{O}(G)}$.

Since we want a Lie algebra action, we take vector space duals to get a map

$$\mathrm{Dist}(G) \to \mathrm{End}_k(\mathcal{F})$$

where $\mathrm{Dist}(G) := \varinjlim (\mathcal{O}(G)/\mathfrak{m}^{i+1})^*$ has the structure of a filtered associative $k$-algebra induced from the Hopf algebra structure of $\mathcal{O}(G)$ (cf. [Jan03, 7.7]). The associativity axiom of $\tau_{\mathcal{F}}$ implies that we actually have a map of $k$-algebras. In particular, as $\mathfrak{g} = (\mathfrak{m}/\mathfrak{m}^2)^* \subset \mathrm{Dist}(G)$, we have a map of Lie algebras $\alpha_{\mathcal{F}} : \mathfrak{g} \to \mathrm{End}_k(\mathcal{F})$, which gives us our desired action.

5.1.10. Once again, we would like to give a more intuitive explanation of this action. Let $k[\epsilon] := k[\epsilon]/\epsilon^2$ denote the dual numbers. We will think of the Lie algebra as $\mathfrak{g} = \mathrm{Der}_k(\mathcal{O}(G), k)$. Observe that we can identify

$$\mathfrak{g} \simeq G_1(k[\epsilon]) \simeq \ker(G(k[\epsilon]) \to G(k)) : \xi \mapsto 1 + \epsilon \cdot \xi.$$

Let $U \overset{j}{\hookrightarrow} X$ be an open affine subset and consider the extended scheme $U[\epsilon] = U \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[\epsilon]$ over $k[\epsilon]$. For $\xi \in \mathfrak{g}$, consider the element $g = 1 + \epsilon \cdot \xi \in G(k[\epsilon])$. We remarked earlier that the discussion of 5.1.6 in fact holds for an arbitrary base, so let us base change from $k$ to $k[\epsilon]$. Then taking local sections of $\tau_{g,j}$ gives an $\mathrm{act}_g^*$-isomorphism $\Gamma(g \cdot U[\epsilon], \mathcal{F}) \to \Gamma(U[\epsilon], \mathcal{F})$. We are working infinitesimally, so $g \mapsto 1 \in G(k)$ implies that $g \cdot U[\epsilon] = U[\epsilon]$. Therefore we in fact have an $\mathrm{act}_g^*$-isomorphism $\Gamma(U, \mathcal{F}) \otimes k[\epsilon] \to \Gamma(U, \mathcal{F}) \otimes k[\epsilon]$ which becomes the identity when we set $\epsilon$ equal to zero. Equivalently, we have a $k[\epsilon]$-linear morphism

$$\Gamma(U, \mathcal{F}) \overset{\Gamma(\tau_{g,j})}{\to} \Gamma(U, \mathcal{F}) \oplus \epsilon \cdot \Gamma(U, \mathcal{F}) : m \mapsto m + \epsilon \cdot \alpha_{\mathcal{F}}(\xi)(m)$$

which is $\mathrm{act}_g^*$-linear. So the Lie algebra action $\alpha_{\mathcal{F}}$ is really just describing the action of $G_1(k[\epsilon])$ points on our quasi-coherent sheaf $\mathcal{F}$.

Applying the above to the case $\mathcal{F} = \mathcal{O}_X$, we have the map $\alpha_{\mathcal{O}} : \mathfrak{g} \to \mathrm{End}_k(\mathcal{O}_X)$. In this case, $\tau_{\mathcal{O}}$ is induced from $\mathrm{act}^*$, so in the notation of the last paragraph, $\mathrm{act}_g^*(f) = f + \epsilon \cdot \alpha_{\mathcal{O}}(\xi)(f)$ for $f \in \mathcal{O}(U)$. Here $\mathrm{act}_g^* : \mathcal{O}(g \cdot U[\epsilon]) = \mathcal{O}(U) \otimes k[\epsilon] \to \mathcal{O}(U) \otimes k[\epsilon]$ is a $k[\epsilon]$-algebra isomorphism. In particular, it respects multiplication so we deduce that $\alpha_{\mathcal{O}}(\xi)$ is a derivation. Therefore $\alpha_{\mathcal{O}}$ is in fact a map of Lie algebras $\mathfrak{g} \to \Gamma(X, \Theta_X)$.

Now for a general $\mathcal{F} \in \mathrm{QCoh}(X, G)$, the $\mathrm{act}_g^*$-linearity condition can be expressed in terms of $\alpha_{\mathcal{O}}$ by the formula

$$\alpha_{\mathcal{F}}(\xi)(f \cdot m) = f \cdot \alpha_{\mathcal{F}}(\xi)(m) + \alpha_{\mathcal{O}}(\xi)(f) \cdot m. \qquad (5.1.10.1)$$

The intuition is that if we base change to $k[\epsilon]$, then we can apply all of the usual techniques and calculations from standard calculus to the algebraic setting.

5.1.11. The map $\alpha_{\mathcal{O}} : \mathfrak{g} \to \Gamma(X, \Theta_X)$ induces an $\mathcal{O}_X$-linear morphism $\sigma : \mathcal{O}_X \otimes_k \mathfrak{g} \to \Theta_X$. Then $\widetilde{\mathfrak{g}}_X := \mathcal{O}_X \otimes_k \mathfrak{g}$ becomes a Lie algebroid when we define the bracket by

$$[f_1 \otimes \xi_1, f_2 \otimes \xi_2] = f_1 f_2 \otimes [\xi_1, \xi_2] + f_1 \alpha_{\mathcal{O}}(\xi_1)(f_2) \otimes \xi_2 - f_2 \alpha_{\mathcal{O}}(\xi_2)(f_1) \otimes \xi_1.$$

The universal enveloping algebra $U(\widetilde{\mathfrak{g}}_X)$ is isomorphic to $\mathcal{O}_X \otimes_k U(\mathfrak{g})$ as an $\mathcal{O}_X$-module.

For an arbitrary $\mathcal{F} \in \mathrm{QCoh}(X, G)$, the formula (5.1.10.1) implies that $\alpha_{\mathcal{F}}$ induces a morphism of $\mathcal{O}_X$-algebras $U(\widetilde{\mathfrak{g}}_X) \to \mathcal{E}nd_k(\mathcal{F})$.

5.1.12. *Principal bundles.* Assume that $\pi : X \to Z$ is a principal $G$-bundle, so that $Z$ represents the quotient stack [Wan11, Remark 2.1.2]. Then the equivalence $\mathrm{QCoh}(G\backslash X) \simeq \mathrm{QCoh}(X, G)$ implies that $\pi^*$ defines an equivalence of categories $\mathrm{QCoh}(Z) \xrightarrow{\sim} \mathrm{QCoh}(X, G)$. The equivariant structure is given by the identification $(\mathrm{act}, \mathrm{pr}_2) : G \times X \xrightarrow{\sim} X \times_Z X$. We would like to describe the inverse functor $\pi_\bullet^G : \mathrm{QCoh}(X, G) \to \mathrm{QCoh}(Z)$. Our notation follows [BL94].

Take $\mathcal{F} \in \mathrm{QCoh}(X, G)$. For an open affine subset $U \subset Z$, we know that $V = \pi^{-1}U$ is an open affine subset of $X$ stable under the action of $G$. By flat descent, we can consider $\mathcal{F}$ as a sheaf in the fpqc topology via pullback. That is to say, we have an equalizer diagram

$$0 \to \Gamma(U, \pi_\bullet^G(\mathcal{F})) \to \Gamma(V, \pi^*\pi_\bullet^G(\mathcal{F})) \rightrightarrows \Gamma(V \underset{U}{\times} V, \mathrm{pr}_1^*\pi^*\pi_\bullet^G(\mathcal{F})).$$

By the definition of $\pi_\bullet^G$ as the inverse, we have $\pi^*\pi_\bullet^G(\mathcal{F}) \simeq \mathcal{F}$. Additionally since $V$ is a $G$-bundle over $U$, we have $(\mathrm{act}, \mathrm{pr}_2) : G \times V \xrightarrow{\sim} V \times_U V$. So it remains for us to determine what the corresponding maps are in the diagram

$$0 \to \Gamma(U, \pi_\bullet^G(\mathcal{F})) \to \Gamma(V, \mathcal{F}) \rightrightarrows \mathcal{O}(G) \underset{k}{\otimes} \Gamma(V, \mathcal{F}).$$

Now note that we are assuming $\pi^*\pi_\bullet^G(\mathcal{F}) \simeq \mathcal{F}$ as $G$-equivariant sheaves, so the isomorphism $\mathrm{pr}_1^*\pi^*\pi_\bullet^G(\mathcal{F}) \simeq \mathrm{pr}_2^*\pi^*\pi_\bullet^G(\mathcal{F})$ corresponds to the $G$-equivariant structure $\mathrm{act}^*\mathcal{F} \simeq \mathrm{pr}_2^*\mathcal{F}$. Therefore the two maps $\Gamma(V, \mathcal{F}) \rightrightarrows \mathcal{O}(G) \otimes \Gamma(V, \mathcal{F})$ correspond to $m \mapsto 1 \otimes m$ and the comodule map described in 5.1.5. Thus $\Gamma(U, \pi_\bullet^G(\mathcal{F}))$ can be described explicitly as the space of $G$-invariant sections $\Gamma(V, \mathcal{F})^G$ under the $G$-action on $\mathcal{F}$. This also explains the notation: we can write $\pi_\bullet^G(\mathcal{F}) = (\pi_\bullet \mathcal{F})^G$.

5.2. **Equivariant twisted differential operators.** In this subsection, we will assume $X$ and $G$ are both smooth[6]. Then the quotient stack $G\backslash X$ is a smooth algebraic $k$-stack, and $X \to G\backslash X$ is a smooth covering.

We define a $G$-equivariant tdo to be the data of a tdo $\mathcal{D}$ on $X$ together with a morphism of $\mathcal{O}_{G \times X}$-algebras $\gamma_{\mathcal{D}} : \mathrm{act}^\sharp \mathcal{D} \to \mathrm{pr}_2^\sharp \mathcal{D}$ such that the diagram (5.1.2.1) commutes, where we replace pullbacks of $\mathcal{O}$-modules with pullbacks of tdo's. The same discussion as for quasi-coherent sheaves shows that this is equivalent to giving a tdo on the quotient stack $G\backslash X$ (see 2.3.3 for the definition of a tdo on an algebraic stack).

5.2.1. Let $\mathcal{D}$ be a $G$-equivariant tdo on $X$. Recall that $\mathcal{D}_G \boxtimes_{\mathcal{O}} \mathcal{D} \xrightarrow{\sim} \mathrm{pr}_2^\sharp \mathcal{D}$ (Example 2.2.6), where $\mathcal{D}_G$ is the sheaf of ordinary differential operators on $G$. Note that $\mathrm{pr}_2^* \mathcal{D} \simeq \mathcal{O}_G \boxtimes_{\mathcal{O}} \mathcal{D}$ naturally embeds into $\mathcal{D}_G \boxtimes_{\mathcal{O}} \mathcal{D}$ as an $\mathcal{O}_{G \times X}$-subalgebra, and it is the centralizer of $\mathrm{pr}_1^{-1}\mathcal{O}_G = \mathcal{O}_G \boxtimes_k k \subset \mathcal{O}_{G \times X}$ in $\mathrm{pr}_2^\sharp \mathcal{D}$. Define the isomorphism

$$\mathrm{exch} = (\mathrm{pr}_1, \mathrm{act}) : G \times X \to G \times X : (g, x) \mapsto (g, gx)$$

[6]In fact, since we are in characteristic 0, Cartier's theorem [DG70, II, §6, 1.1] implies that $G$ is automatically smooth.

and observe that $\mathrm{act} = \mathrm{pr}_2 \circ \mathrm{exch}$. Since exch is an isomorphism, the pullback $\mathrm{exch}^*$ simply changes the action of the ring $\mathcal{O}_{G\times X}$, and we have $\mathrm{act}^\sharp \mathcal{D} \simeq \mathrm{exch}^* \mathrm{pr}_2^\sharp \mathcal{D}$. Therefore $\gamma_{\mathcal{D}}$ is a morphism of $\mathcal{O}_{G\times X}$-algebras

$$\mathcal{O}_{G\times X} \underset{\mathrm{exch}^{-1}(\mathcal{O}_{G\times X})}{\otimes} \mathrm{exch}^{-1}(\mathrm{pr}_2^\sharp \mathcal{D}) \xrightarrow{\sim} \mathrm{pr}_2^\sharp \mathcal{D}.$$

We note the similarity of this setup with the one described in 5.1.6.

Now since $\mathrm{pr}_1 = \mathrm{pr}_1 \circ \mathrm{exch}$, we see that $\mathrm{pr}_1^{-1}\mathcal{O}_G$ naturally lives in both sides. The $\mathrm{pr}_1^{-1}\mathcal{O}_G$-centralizer on the left is $\mathrm{exch}^*\mathrm{pr}_2^*\mathcal{D} \simeq \mathrm{act}^*\mathcal{D}$, so $\gamma_{\mathcal{D}}$ induces an isomorphism

$$\tau_{\mathcal{D}} : \mathrm{act}^*\mathcal{D} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{D}.$$

Since the embedding $\mathrm{pr}_2^*\mathcal{D} \subset \mathrm{pr}_2^\sharp\mathcal{D}$ is right inverse to the action of $\mathrm{pr}_2^\sharp\mathcal{D}$ on $1\otimes 1 \in \mathrm{pr}_2^*\mathcal{D}$, we see that $\tau_{\mathcal{D}}$ is a $\gamma_{\mathcal{D}}$-isomorphism with $\tau_{\mathcal{D}}(1\otimes 1) = 1 \otimes 1$.

If we consider one of the projections $\mathrm{pr}_{ij} : G\times G\times X \to G\times X$, then $\mathrm{pr}_{ij}^*(\tau_{\mathcal{D}})$ is naturally a $\mathrm{pr}_{ij}^\sharp(\gamma_{\mathcal{D}})$-morphism sending $1\otimes 1\otimes 1 \mapsto 1\otimes 1\otimes 1$. Therefore $\tau_{\mathcal{D}}$ satisfies the cocycle condition since $\gamma_{\mathcal{D}}$ does. This gives $\mathcal{D}$ the structure of a $G$-equivariant quasi-coherent sheaf. Consequently, we get a Lie algebra action $\alpha_{\mathcal{D}} : \mathfrak{g} \to \mathrm{End}(\mathcal{D})$.

**Example 5.2.2.** Let $\mathcal{L}$ be a $G$-equivariant line bundle on $X$. Then

$$\gamma_{\mathcal{D}_{\mathcal{L}}} := (\tau_{\mathcal{L}} \circ \bullet \circ \tau_{\mathcal{L}}^{-1}) : \mathcal{D}_{\mathrm{act}^*\mathcal{L}} \xrightarrow{\sim} \mathcal{D}_{\mathrm{pr}_2^*\mathcal{L}}$$

makes $\mathcal{D}_{\mathcal{L}}$ a $G$-equivariant tdo by Example 2.2.4. Since $\mathcal{D}_{\mathcal{L}}$ is then also $G$-equivariant as a quasi-coherent sheaf, we have a Lie algebra action $\alpha_{\mathcal{D}_{\mathcal{L}}} : \mathfrak{g} \to \mathrm{End}_k(\mathcal{D}_{\mathcal{L}})$. On the other hand, we have the action $\alpha_{\mathcal{L}} : \mathfrak{g} \to \mathrm{End}_k(\mathcal{L})$ coming from the $G$-equivariance of $\mathcal{L}$. Since $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_X$ and we know $\alpha_{\mathcal{O}}$ lands in $\Gamma(X, \Theta_X)$, we deduce from (5.1.10.1) that $\alpha_{\mathcal{L}}$ sends $\mathfrak{g} \to \Gamma(X, F_1\mathcal{D}_{\mathcal{L}})$. We claim that $\alpha_{\mathcal{D}_{\mathcal{L}}} = \mathrm{ad}\,\alpha_{\mathcal{L}} = [\alpha_{\mathcal{L}}, \bullet]$. To see this, fix $\xi \in \mathfrak{g}$ and pull back $\tau_{\mathcal{D}_{\mathcal{L}}}$ along $1 + \epsilon\cdot\xi : \mathrm{Spec}\, k[\epsilon] \to G$ so we are considering sheaves on $X[\epsilon]$. Since $\tau_{\mathcal{D}_{\mathcal{L}}}$ is defined as conjugation by $\tau_{\mathcal{L}}$, the standard calculus computation over $k[\epsilon]$ proves the claim.

**Example 5.2.3.** By the previous example and Example 5.1.8, we have a $G$-equivariant structure on $\mathcal{D}_X$. Let us give an explicit description of the $G$-action on $\Gamma(X, \mathcal{D}_X)$. Take a differential operator $\partial \in \Gamma(X, \mathcal{D}_X) = \mathcal{D}if(\mathcal{O}_X, \mathcal{O}_X)$ and a geometric point $g \in G(k)$. Since $\tau_{\mathcal{D}_X} = (\tau_{\mathcal{O}} \circ \bullet \circ \tau_{\mathcal{O}}^{-1})$, we see that the new operator $g\cdot\partial$ sends a function $f \in \Gamma(X, \mathcal{O}_X)$ to the regular function $x \mapsto (g\cdot\partial(g^{-1}\cdot f))(x) = \partial(g^{-1}\cdot f)(g^{-1}x)$.

5.2.4. *Differential operators on $G$.* Let us provide some motivation for why the Lie algebra will play such a central role in our future discussions. Consider the commuting actions of $G$ on itself by left and right translations. These give Lie algebra morphisms

$$\alpha^\ell : \mathfrak{g} \to \Gamma(G, \mathcal{D}_G) \leftarrow \mathfrak{g} : \alpha^r$$

whose images commute. In addition, the image of $\alpha^\ell_{\mathcal{O}}$ is $G$-invariant with respect to the right $G$-equivariant structure and vice versa. This can be seen at the level of $G(S[\epsilon])$-points or simply by noting that the left (resp. right) action comes from evaluating $\mathrm{mult}^* : \mathcal{O}(G) \to \mathcal{O}(G)\otimes\mathcal{O}(G)$ on the first (resp. second) coordinate.

Before proceeding, let us check that these morphisms are the standard ones as described in, e.g., [Hum75, 9.2]. Take $\xi \in \mathfrak{g}$, a function $f \in \Gamma(G, \mathcal{O}_G)$, and geometric points $g, x \in G(k)$. Let $\Delta : \Gamma(G, \mathcal{O}_G) \to \mathcal{O}(G) \otimes \Gamma(G, \mathcal{O}_G)$ be the comodule map with respect to right translations; so $\Delta(f)(g, x) = f(xg)$. By definition, $\alpha^r(\xi)(f)$ is the function sending $x \mapsto \xi(f(x\bullet)) = \xi(x^{-1}\cdot f)$, which is the usual map that associates a left invariant vector field to $\xi$.

Now consider the induced morphisms of $\mathcal{O}_G$-algebras

$$\alpha^\ell : U(\widetilde{\mathfrak{g}}^\ell_G) \to \mathcal{D}_G \leftarrow U(\widetilde{\mathfrak{g}}^r_G) : \alpha^r.$$

**Proposition 5.2.5.** *The maps $\alpha^\ell$ and $\alpha^r$ are isomorphisms.*

*Proof.* We will deal with the left case. By taking the associated graded modules, we have a map $\mathcal{O}_G \otimes_k \operatorname{gr} U(\mathfrak{g}) \to \operatorname{Sym}_{\mathcal{O}_G}(\Theta_G)$. Since the composition

$$\mathcal{O}_G \underset{k}{\otimes} \operatorname{Sym}_k(\mathfrak{g}) \twoheadrightarrow \mathcal{O}_G \underset{k}{\otimes} \operatorname{gr} U(\mathfrak{g}) \to \operatorname{Sym}_{\mathcal{O}_G}(\Theta_G)$$

is an isomorphism, we deduce that both of the intermediate arrows are also isomorphisms. In particular, $\operatorname{Sym}(\mathfrak{g}) \simeq \operatorname{gr} U(\mathfrak{g})$, so we have proved the PBW theorem as a bonus. □

**Corollary 5.2.6.** *We have canonical isomorphisms $U(\mathfrak{g}) \simeq \kappa(1) \otimes_{\mathcal{O}_G} \mathcal{D}_G \simeq \operatorname{Dist}(G)$.*

We therefore see that in characteristic 0, the Lie algebra completely determines $\operatorname{Dist}(G)$, so we may focus our attention on it. This is not the case in positive characteristic, where the Lie algebra plays less of a role and we do need to consider all infinitesimal neighborhoods (cf. [Jan03, Chapter 7]).

Proposition 5.2.5 says that the differential operators $\mathcal{D}_G$ are determined by $\mathfrak{g}$ through the universal enveloping algebra of the Lie algebroid corresponding to either $\alpha^\ell$ or $\alpha^r$. Since $\operatorname{pr}_2^\sharp \mathcal{D} \simeq \mathcal{D}_G \boxtimes_{\mathcal{O}} \mathcal{D}$, this motivates us to find an alternative characterization of $G$-equivariant tdo's in terms of Lie algebra actions.

5.2.7. Let $\mathcal{D}$ be a $G$-equivariant tdo on $X$ and put $\mathcal{L} = \operatorname{Lie} \mathcal{D}$. Since $\operatorname{exch} = (\operatorname{pr}_1, \operatorname{act})$ is an isomorphism, the canonical map $(d\operatorname{pr}_1, d\operatorname{act}) : \Theta_{G \times X} \to \operatorname{pr}_1^* \Theta_G \times \operatorname{act}^* \Theta_X$ is an isomorphism of $\mathcal{O}_{G \times X}$-modules. Therefore $\operatorname{act}^\sharp \mathcal{L} \simeq \operatorname{pr}_1^* \Theta_G \times \operatorname{act}^* \mathcal{L}$ as $\mathcal{O}_{G \times X}$-modules, and the anchor corresponds to the composition $\operatorname{pr}_1^* \Theta_G \times \operatorname{act}^* \mathcal{L} \to \operatorname{pr}_1^* \Theta_G \times \operatorname{act}^* \Theta_X \xleftarrow{\sim} \Theta_{G \times X}$. Recall from Proposition 2.2.1 that $\operatorname{Lie}(\operatorname{act}^\sharp \mathcal{D}) \simeq \operatorname{act}^\sharp \mathcal{L}$. Therefore $\gamma_{\mathcal{D}}$ is determined by the restriction $\gamma : \operatorname{act}^\sharp \mathcal{L} \to \operatorname{pr}_2^\sharp$, which is a morphism of Lie algebroids. By the above, this corresponds to

$$\gamma : \operatorname{pr}_1^* \Theta_G \times \operatorname{act}^* \mathcal{L} \to \operatorname{pr}_1^* \Theta_G \times \operatorname{pr}_2^* \mathcal{L} \tag{5.2.7.1}$$

over $\Theta_{G \times X}$. For $\xi \in \mathfrak{g}$, we then have $\gamma(\alpha^r(\xi), 0) = (\alpha^r(\xi), i_{\mathfrak{g}}(\xi))$ where $i_{\mathfrak{g}}(\xi)$ is a global section of $\Gamma(G \times X, \operatorname{pr}_2^* \mathcal{L}) \simeq \mathcal{O}(G) \otimes_k \Gamma(X, \mathcal{L})$. We claim that $i_{\mathfrak{g}}(\xi)$ in fact lies in $\Gamma(X, \mathcal{L})$. The proof is a careful inspection of the cocycle condition for $\gamma$ which is rather technical and may be skipped by the reader.

*Proof.* For the sake of readability, let $g_1, g_2 : G \times G \times X \to G$ and $x : G \times G \times X \to X$ denote the projections. Then for example $\operatorname{id} \times \operatorname{act} = (g_1, g_2 x)$. The cocycle condition then requires

$$\begin{tikzcd}
(g_1, g_2 x)^\sharp \operatorname{act}^\sharp \mathcal{L} \arrow[r, "(g_1, g_2 x)^\sharp(\gamma)"] \arrow[d, "\sim"'] & (g_1, g_2 x)^\sharp \operatorname{pr}_2^\sharp \mathcal{L} \arrow[r, "\sim"] & (g_2, x)^\sharp \operatorname{act}^\sharp \mathcal{L} \arrow[d, "(g_2, x)^\sharp(\gamma)"] \\
(g_1 g_2, x)^\sharp \operatorname{act}^\sharp \mathcal{L} \arrow[r, "(g_1 g_2, x)^\sharp(\gamma)"] & (g_1 g_2, x)^\sharp \operatorname{pr}_2^\sharp \mathcal{L} \arrow[r, "\sim"] & (g_1 g_2, x) \operatorname{pr}_2^\sharp \mathcal{L}
\end{tikzcd}$$

to commute. Observe that $(g_1, g_2 x)^* (\operatorname{act}^\sharp \mathcal{L}) \simeq (g_1, g_2 x)^* (\operatorname{pr}_1^* \Theta_G \times \operatorname{act}^* \mathcal{L}) \simeq g_1^* \Theta_G \times (g_1 g_2 x)^* \mathcal{L}$. We also have Cartesian squares

$$\begin{tikzcd}
g_2^* \Theta_G \times g_1^* \Theta_G \times (g_1 g_2 x)^* \mathcal{L} \arrow[r] \arrow[d, "\operatorname{pr}_{23}"'] & g_2^* \Theta_G \times g_1^* \Theta_G \times (g_1 g_2 x)^* \Theta_X \arrow[d, "\operatorname{pr}_{23}"'] & \Theta_{G \times G \times X} \arrow[l, "\sim"'] \arrow[d, "d(g_1, g_2 x)"] \\
g_1^* \Theta_G \times (g_1 g_2 x)^* \mathcal{L} \arrow[r] & (g_1, g_2 x)^* (\operatorname{pr}_1^* \Theta_G \times \operatorname{act}^* \Theta_X) & (g_1, g_2 x)^* \Theta_{G \times X} \arrow[l, "\sim"']
\end{tikzcd}$$

Hence $(g_1, g_2x)^\sharp \mathrm{act}^\sharp \mathcal{L} \simeq g_2^*\Theta_G \times g_1^*\Theta_G \times (g_1g_2x)^*\mathcal{L}$ with anchor given by the top row of the above diagram. Similar considerations for the other entries give us a reformulation of the cocycle condition in terms of more familiar objects:

$$\begin{tikzcd}
g_2^*\Theta_G \times g_1^*\Theta_G \times (g_1g_2x)^*\mathcal{L} \arrow[r, "\mathbf{a}"] \arrow[d, "\mathbf{d}"'] & g_2^*\Theta_G \times g_1^*\Theta_G \times (g_2x)^*\mathcal{L} \arrow[r, "\mathbf{b}"] & g_1^*\Theta_G \times g_2^*\Theta_G \times (g_2x)^*\mathcal{L} \arrow[d, "\mathbf{c}"] \\
g_2^*\Theta_G \times (g_1g_2)^*\Theta_G \times (g_1g_2x)^*\mathcal{L} \arrow[r, "\mathbf{e}"] & g_2^*\Theta_G \times (g_1g_2)^*\Theta_G \times x^*\mathcal{L} \arrow[r, "\mathbf{f}"] & g_1^*\Theta_G \times g_2^*\Theta_G \times x^*\mathcal{L}
\end{tikzcd}$$

The morphisms $\mathbf{a} = \mathrm{id} \times (g_1, g_2x)^*(\gamma)$, $\mathbf{c} = \mathrm{id} \times (g_2, x)^*(\gamma)$, and $\mathbf{e} = \mathrm{id} \times (g_1g_2, x)^*(\gamma)$ are all induced from $\gamma$. The transition isomorphism $\mathbf{b}$ is a morphism of Lie algebroids compatible with anchors, so it must swap the $g_i^*\Theta_G$ for $i = 1, 2$ while fixing $(g_2x)^*\mathcal{L}$. The morphism $\mathbf{d}$ is the product of $g_2^*\Theta_G \times g_1^*\Theta_G \simeq g_2^*\Theta_G \times (g_1g_2)^*\Theta_G$ with $\mathrm{id}_{(g_1g_2x)^*\mathcal{L}}$. Let us describe the first of these. Define $p_1, p_2$ and $\mu : G \times G \to G$ to be the projections and the action $\mu(g_1, g_2) = g_2g_1^{-1}$. Then $g_1^*\Theta_G \to (g_1g_2)^*\Theta_G$ is the pullback $(g_2, g_1g_2)^*(\mu^*\Theta_G \to p_2^*\Theta_G)$ of the $G$-equivariant structure on $\Theta_G$ induced by $\mu$. Let $\Delta' : \mathfrak{g} \to \mathcal{O}(G) \otimes_k \mathfrak{g}$ denote the comodule map corresponding to the adjoint representation. For $\eta \in \mathfrak{g}$, write $\Delta'(\eta) = \Sigma f_i \otimes \eta_i'$ for $f_i \in \mathcal{O}(G)$ and $\eta_i' \in \mathfrak{g}$. Since the $\alpha^r(\eta_i')$ are left invariant vector fields, we deduce that

$$\mathbf{d}(0, g_1^*(\alpha^r(\eta)), 0) = (0, \Sigma\, g_2^*(f_i) \cdot (g_1g_2)^*(\alpha^r(\eta_i')), 0). \tag{5.2.7.2}$$

Similarly by left invariance, we see that $\mathbf{d}(g_2^*(\alpha^r(\xi)), 0, 0) = (g_2^*(\alpha^r(\xi)), (g_1g_2)^*(\alpha^r(\xi)), 0)$. Lastly $\mathbf{f}$ fixes the third coordinate and equals the inverse of $\mathbf{d}$ followed by a swap on the first two coordinates. Hence the commutativity of the diagram implies that

$$\begin{aligned}
(0, g_2^*(\alpha^r(\xi)), (g_2, x)^*(i_{\mathfrak{g}}(\xi)) &= \mathbf{cba}(g_2^*(\alpha^r(\xi)), 0, 0) \\
&= \mathbf{fed}(g_2^*(\alpha^r(\xi), 0, 0)) = (0, g_2^*(\alpha^r(\xi)), (g_1g_2, x)^*(i_{\mathfrak{g}}(\xi))).
\end{aligned}$$

The equality $(g_2, x)^*(i_{\mathfrak{g}}(\xi)) = (g_1g_2, x)^*(i_{\mathfrak{g}}(\xi))$ implies that $i_{\mathfrak{g}}(\xi) \in \Gamma(X, \mathcal{L})$ by flat descent. □

As a consequence, $\mathrm{pr}_1^*\Theta_G$ acts trivially on $i_{\mathfrak{g}}(\xi)$. Since $\gamma$ is a morphism of Lie algebroids, we conclude that $i_{\mathfrak{g}} : \mathfrak{g} \to \Gamma(X, \mathcal{L}) \subset \Gamma(X, \mathcal{D})$ is a Lie algebra map. Note that $\gamma$ restricts to $\tau_{\mathcal{D}}$ on $\mathrm{act}^*\mathcal{L} \to \mathrm{pr}_2^*\mathcal{L}$. Since $\mathcal{D}_G \simeq U(\widetilde{\mathfrak{g}}_G^r)$ by Proposition 5.2.5, we see that $\gamma_{\mathcal{D}}$ is determined by $\tau_{\mathcal{D}}$ and $i_{\mathfrak{g}}$. The next lemma makes this precise.

**Lemma 5.2.8.** *A tdo on the quotient stack $G\backslash X$ is the same as a tdo $\mathcal{D}$ on $X$ equipped with a pair $(\tau_{\mathcal{D}}, i_{\mathfrak{g}})$, where $\tau_{\mathcal{D}} : \mathrm{act}^*\mathcal{D} \to \mathrm{pr}_2^*\mathcal{D}$ is a $G$-equivariant structure on $\mathcal{D}$ as a left (or right) $\mathcal{O}_X$-module and $i_{\mathfrak{g}} : \mathfrak{g} \to \Gamma(X, \mathcal{D})$ is a Lie algebra map such that*

(i) *$\tau_{\mathcal{D}}$ is a ring homomorphism between the subrings of $\mathrm{pr}_1^{-1}\mathcal{O}_G$-centralizers.*
(ii) *$i_{\mathfrak{g}}$ is a morphism of $G$-modules (where $\mathfrak{g}$ has the adjoint representation).*
(iii) *The Lie algebra action $\alpha_{\mathcal{D}} : \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ induced by $\tau_{\mathcal{D}}$ coincides with $\mathrm{ad}\, i_{\mathfrak{g}}$.*

*Proof.* Cf. [BB93, Lemma 1.8.7(i)]. Consider a tdo on $G\backslash X$. This is the same as a $G$-equivariant tdo $\mathcal{D}$ on $X$. Given $\gamma_{\mathcal{D}}$, we have constructed $\tau_{\mathcal{D}}$ and $i_{\mathfrak{g}}$ in the previous subsections. Since $\gamma_{\mathcal{D}}$ is a map of $\mathcal{O}_{G\times X}$-algebras, so is $\tau_{\mathcal{D}}$. Let us momentarily adopt the notation of the previous proof, where $\Delta' : \mathfrak{g} \to \mathcal{O}(G) \otimes_k \mathfrak{g}$ is the comodule map of the adjoint representation and $\Delta'(\eta) = \Sigma f_i \otimes \eta_i'$ for $\eta \in \mathfrak{g}$. Using (5.2.7.2), the cocycle condition tells us that

$$\begin{aligned}
(g_1^*(\alpha^r(\eta)), 0, (g_2, x)^*(\tau_{\mathcal{D}})((g_2x)^*(i_{\mathfrak{g}}(\eta)))) &= \mathbf{cba}(0, g_1^*(\alpha^r(\eta)), 0) = \mathbf{fed}(0, g_1^*(\alpha^r(\eta)), 0) \\
&= (g_1^*(\alpha^r(\eta)), 0, \Sigma\, g_2^*(f_i) \cdot x^*(i_{\mathfrak{g}}(\eta_i')))
\end{aligned}$$

where we are considering $i_{\mathfrak{g}}(\eta) \in \mathcal{D}$. Looking at the third coordinate, we have

$$(g_2, x)^*\big(\tau_{\mathcal{D}}(\mathrm{act}^*(i_{\mathfrak{g}}(\eta)))\big) = (g_2, x)^*\big(\Sigma\, \mathrm{pr}_1^*(f_i) \cdot \mathrm{pr}_2^*(i_{\mathfrak{g}}(\eta_i'))\big).$$

Since $(g_2, x)^*$ is injective by flat descent, this implies (ii). For $\xi \in \mathfrak{g}$ and $P \in \mathcal{D}$, we have $\gamma_{\mathcal{D}}(\mathrm{exch}^*[\mathrm{pr}_1^*(\alpha^r(\xi)), \mathrm{pr}_2^* P]) = [\mathrm{pr}_1^*(\alpha^r(\xi)) + \mathrm{pr}_2^*(i_{\mathfrak{g}}(\xi)), \tau_{\mathcal{D}}(\mathrm{act}^* P)] = 0$ since $\gamma_{\mathcal{D}}$ is a ring homomorphism. Write $\tau_{\mathcal{D}}(\mathrm{act}^* P) = \Sigma f_i \otimes Q_i$ for $f_i \in \mathrm{pr}_1^{-1}\mathcal{O}_G$ and $Q_i \in \mathrm{pr}_2^{-1}\mathcal{D}$. Evaluating the RHS, we have $-\Sigma \alpha^r(\xi)(f_i) \otimes Q_i = \Sigma f_i \otimes [i_{\mathfrak{g}}(\xi), Q_i]$. Taking the fiber at $1 \in G$, we get $\alpha_{\mathcal{D}}(\xi)(P) = -\Sigma \xi(f_i) Q_i = (\mathrm{ad}\, i_{\mathfrak{g}}(\xi))(P)$. This proves (iii).

Conversely, suppose we have a tdo $\mathcal{D}$ on $X$ with a pair $(\tau_{\mathcal{D}}, i_{\mathfrak{g}})$ satisfying (i)-(iii). Since $\mathrm{act}^\sharp \mathcal{D} \simeq \mathrm{exch}^*(\mathcal{D}_G \boxtimes_{\mathcal{O}} \mathcal{D}) \simeq \mathrm{exch}^*(U(\widetilde{\mathfrak{g}}_G^r) \boxtimes_{\mathcal{O}} \mathcal{D})$, we can define a map of $\mathcal{O}_{G\times X}$-modules $\gamma_{\mathcal{D}} : \mathrm{act}^\sharp \mathcal{D} \to \mathrm{pr}_2^\sharp \mathcal{D}$ in terms of $\tau_{\mathcal{D}}$ and $i_{\mathfrak{g}}$. This will be an algebra homomorphism due to the commutator relations that arise from $i_{\mathfrak{g}}$ being a Lie algebra map together with conditions (i) and (iii). Our previous discussions show that $\gamma_{\mathcal{D}}$ satisfies the cocycle condition since $\tau_{\mathcal{D}}$ does and $i_{\mathfrak{g}}$ lands in $\Gamma(X, \mathcal{D})$ and satisfies (ii). Therefore we have made $\mathcal{D}$ into a $G$-equivariant tdo. Clearly the above constructions are mutually inverse. $\square$

5.2.9. *Equivariant $\mathcal{D}$-modules.* Let $\mathcal{D}, \tau_{\mathcal{D}}$ be a $G$-equivariant tdo on $X$, which also defines a tdo on the quotient stack $G\backslash X$. We define a $G$-equivariant $\mathcal{D}$-module on $X$ to be a $\mathcal{D}$-module $\mathcal{M}$ together with a $\tau_{\mathcal{D}}$-isomorphism $\tau_{\mathcal{M}} : \mathrm{act}^*\mathcal{M} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{M}$ such that $\mathcal{M}$ is $G$-equivariant as a quasi-coherent sheaf. From 2.3.3, this is equivalent to defining a $\mathcal{D}$-module on $G\backslash X$. Here we recall that $\mathrm{act}^*\mathcal{M}$, $\mathrm{pr}_2^*\mathcal{M}$ are modules over the tdo's $\mathrm{act}^\sharp\mathcal{D}$, $\mathrm{pr}_2^\sharp\mathcal{D}$ respectively, and a $\tau_{\mathcal{D}}$-morphism is a morphism of $\mathrm{pr}_2^\sharp\mathcal{D}$-modules where $\mathrm{pr}_2^\sharp\mathcal{D}$ acts on $\mathrm{act}^*\mathcal{M}$ through $\tau_{\mathcal{D}}^{-1}$. Consider $\mathrm{pr}_2^*\mathcal{D} \subset \mathrm{pr}_2^\sharp\mathcal{D}$ as a subalgebra. If $\tau_{\mathcal{M}}$ is only $\mathrm{pr}_2^*\mathcal{D}$-linear in the definition above, then we say that $\mathcal{M}$ is a *weakly* $G$-equivariant $\mathcal{D}$-module.

**Example 5.2.10.** Suppose $X = \mathrm{Spec}\, k$ is a point and take a character $i_{\mathfrak{g}} \in \mathfrak{g}^*$ defining a $G$-equivariant tdo $\mathcal{D}$ as in Example **??**. A weakly $G$-equivariant $\mathcal{D}$-module is simply a $G$-representation $V$. In order for $V$ to be $G$-equivariant, we require $\tau_V : \mathcal{O}_G \otimes V \to \mathcal{O}_G \otimes V$ to be a morphism of $\mathcal{D}_G$-modules. This is equivalent to requiring $\alpha_V = i_{\mathfrak{g}}$.

5.3. **Quotients by subgroups.** Let $H \subset G$ be a closed subgroup. Then the quotient stack $G/H$ is representable by a quasi-projective $G$-scheme $X$, so $\pi : G \to X$ is a (right) principal $H$-bundle [DG70, III, §3, Theorem 5.4]. One has an isomorphism of quotient stacks $(\cdot/H) \xrightarrow{\sim} G\backslash X$ sending an $H$-bundle $\mathcal{P}$ to the twist $\mathcal{P} \overset{H}{\times} G$ (see [Wan11, Lemmas 2.1.1, 2.4.2]). As a consequence, we have the following equivalences:

$$\mathrm{QCoh}(X, G) \xleftarrow{\sim} \mathrm{QCoh}(G\backslash X) \xrightarrow{\sim} \mathrm{QCoh}(\cdot/H) \xrightarrow{\sim} \mathrm{Rep}(H).$$

**Lemma 5.3.1.** *There is an equivalence of categories between (left) $G$-equivariant sheaves on $X$ and (right) $H$-representations defined by taking the fiber at the point $\pi(1) \in X(k)$.*

*Proof.* If we start with the trivial bundle $H$ over $k$, then the twist by $G$ is just the trivial bundle $G$ over $k$. Therefore we have the following 2-commutative diagram

$$\begin{tikzcd} G \times X \arrow[r, shift left, "\mathrm{act}"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r, "G \times X \to X"] & G\backslash X \\ H \arrow[u, "h \mapsto (h, \pi(1))"] \arrow[r, shift left] \arrow[r, shift right] & \cdot \arrow[u, "\pi(1)"'] \arrow[r, "H"] & \cdot/H \arrow[u, "\sim"'] \end{tikzcd}$$

Consider $\mathcal{F} \in \mathrm{QCoh}(X, G)$. From the definitions, it follows that the $H$-equivariant sheaf on the point is defined to be the pullback of $\mathcal{F}$ along $\{\pi(1)\} \hookrightarrow X$, i.e., the fiber $\mathcal{F} \otimes_{\mathcal{O}_X} \kappa(\pi(1))$. We also see that the $H$-equivariant structure is gotten by pulling back $\tau_{\mathcal{F}}$ along the map $H \to G \times X$ defined in the diagram above. $\square$

The inverse functor $\mathcal{L}(\bullet) : \mathrm{Rep}(H) \to \mathrm{QCoh}(X, G)$ can be constructed explicitly as $\mathcal{L}(V) = (\pi_\bullet \mathcal{O}_G \otimes V)^H$, i.e., for an open subset $U \subset X$,

$$\Gamma(U, \mathcal{L}(V)) = \{f \in \Gamma(\pi^{-1}U, \mathcal{O}_G) \otimes V \mid f(gh) = h^{-1}f(g) \text{ for } g \in \pi^{-1}U, h \in H\},$$

where we are considering the action on sections of $\mathcal{O}_G$ induced from the $G$-equivariant structure of right translations. This is a sheaf, and we can give it a $G$-equivariant structure by left translations (cf. [Jan03, 5.8]).

5.3.2. Using the same isomorphisms of stacks as before, we also have from our previous discussions the following equivalence for tdo's.

**Theorem 5.3.3.** *The $G$-equivariant tdo's on $X$ are classified by $\lambda \in \mathfrak{h}^*$ such that $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$.*

Given $\lambda$, the corresponding tdo can be explicitly constructed as

$$\mathcal{A}_X(\lambda) := U(\widetilde{\mathfrak{g}}_X) / \sum_{\xi \in \widetilde{\mathfrak{g}}_X} U(\widetilde{\mathfrak{g}}_X)(\xi - \lambda(\xi)).$$

The $G$-equivariant $\mathcal{A}_X(\lambda)$-modules on $X$ are then equivalent to $H$-modules where $\mathfrak{h}$ acts by $\lambda$.

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