ZARISKI'S MAIN THEOREM

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In this writeup we prove the theorem on formal functions and use it to deduce Zariski's Main Theorem, Stein factorization, and further corollaries.

1. The Theorem on Formal Functions

1.1. **Inverse limits.** Let X be a topological space and R_X a sheaf of rings on it. Then inverse limits exist in the category $Sh(R_X\text{-mod})$. If $(\mathscr{F}_n)$ is an inverse system, define $\mathscr{F} = \varprojlim \mathscr{F}_n$ by $\Gamma(U, \mathscr{F}) = \varprojlim \Gamma(U, \mathscr{F}_n)$ in the category of modules. Then $\mathscr{F}$ is in fact a sheaf of R_X-modules and satisfies the universal property. See [Har77, II.9.2] for details.

1.2. We consider the following situation. Let $f : X \to Y$ be a proper map between Noetherian schemes, $\mathscr{I} \subset \mathscr{O}_Y$ a sheaf of ideals, and $\mathscr{F}$ a coherent sheaf on X. For $n \geq 1$ let $i_n : Y_n \hookrightarrow Y$ be the closed embedding corresponding to $\mathscr{I}^n$. Consider the fibered product diagram

$$\begin{tikzcd} X_n \arrow[r, "i'_n"] \arrow[d, "f_n"] & X \arrow[d, "f"] \\ Y_n \arrow[r, "i_n"] & Y \end{tikzcd}$$

By base change, i'_n is also a closed embedding, hence affine. We have the unit map $\mathscr{F} \to i'_{n*} i'^*_n(\mathscr{F})$. Applying Rf_* gives a map $Rf_*(\mathscr{F}) \to Rf_* i'_{n*} i'^*_n(\mathscr{F})$. Since i'_n is affine, $Ri'_{n*} \simeq i'_{n*}$. By Leray's spectral sequence,

$$Rf_* i'_{n*} \simeq R(f \circ i'_n)_* \simeq R(i_n \circ f_n)_* \simeq i_{n*} Rf_{n*}.$$

Applying H^k, we have a map $R^k f_*(\mathscr{F}) \to R^k f_* i'_{n*} i'^*_n(\mathscr{F}) \simeq i_{n*} R^k f_{n*}(i'^*_n(\mathscr{F}))$. Applying $i_{n*} i^*_n$ to both sides and using $i^*_n i_{n*} \simeq \mathrm{id}$, we have a map

$$i_{n*} i^*_n R^k f_*(\mathscr{F}) \to R^k f_*(\mathscr{F}_n)$$

where $\mathscr{F}_n := i'_{n*} i'^*_n(\mathscr{F})$ is coherent on X. Since f is proper, $R^k f_*(\mathscr{F}), R^k f_*(\mathscr{F}_n)$ are coherent sheaves on Y for all $k \geq 0$. With a slight abuse of notation, we can write the above map more intuitively as

$$R^k f_*(\mathscr{F}) \underset{\mathscr{O}_Y}{\otimes} (\mathscr{O}_Y / \mathscr{I}^n) \to R^k f_*(\mathscr{F}_n), \qquad \mathscr{F}_n = \mathscr{F} / \mathscr{I}^n \mathscr{F}.$$

For $n \geq 1$, both sides form inverse systems of sheaves, since i_{n+1} factors through i_n, which implies i'_{n+1} factors through i'_n. It also follows that the maps are compatible with inverse systems. Taking inverse limits, we have a map

$$\widehat{R^k f_*(\mathscr{F})} \to \varprojlim_n R^k f_*(\mathscr{F}_n) \tag{1}$$

since the left hand side is just the completion.

Theorem 1.2.1 (Theorem on Formal Functions)**.** *Let $f : X \to Y$ be a proper map of Noetherian schemes, $\mathscr{I} \subset \mathscr{O}_Y$ a sheaf of ideals, and $\mathscr{F}$ a coherent sheaf on X. Then the natural map in* (1) *is an isomorphism, for all $k \geq 0$.*

For the proof of Theorem 1.2.1, we mainly follow [sta, 21.16], though [Gro61, Ch. III, 4.1-2] and [Har77, III.11] were also used as references.

1.3. Assume the same conditions as in 1.2. Additionally assume $Y = \operatorname{Spec} A$ is affine, so $\mathscr{I} = \operatorname{Loc}(I)$ for an ideal $I \subset A$. Then for $n \geq 0$, let $I^n\mathscr{F}$ be the image of $f^\bullet(\mathscr{I}^n) \otimes_{f^\bullet(\mathscr{O}_Y)} \mathscr{F} \simeq f^*(\mathscr{I}^n) \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{F}$. An element $x \in I^m$ defines via multiplication a map $I^n\mathscr{F} \to I^{n+m}\mathscr{F}$. By functoriality of $R\Gamma_X$, we have a map

$$H^k(X, I^n\mathscr{F}) \to H^k(X, I^{n+m}\mathscr{F}),$$

which gives $\bigoplus_{n\geq 0} H^k(X, I^n\mathscr{F})$ the structure of a graded $S = \bigoplus_{n\geq 0} I^n$-module.

Lemma 1.3.1. *Under the assumptions of 1.3, $\bigoplus_{n\geq 0} H^k(X, I^n\mathscr{F})$ is a finitely generated S-module.*

Proof. Consider the fibered product diagram

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

By base change, we know f' is proper and π is affine. Since $\mathscr{F}$ is coherent on X, we have $\pi^*\mathscr{F}$ is coherent on X'. Thus $R^k f'_*(\pi^*\mathscr{F})$ is coherent on S. Since A is Noetherian and S is a finitely generated A-algebra, S is also Noetherian. It follows that $R^k f'_*(\pi^*\mathscr{F}) = \operatorname{Loc} H^k(X', \pi^*\mathscr{F})$ (cf. [Har77, III.8.5]). Therefore $H^k(X', \pi^*\mathscr{F})$ is finitely generated as an S-module. As π is affine, $R\pi_* \simeq \pi_*$ and by Leray's spectral sequence, $H^k(X', \pi^*\mathscr{F}) \simeq H^k(X, \pi_*\pi^*\mathscr{F})$. By the projection formula,

$$\pi_*\pi^*\mathscr{F} \simeq \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \pi_*(\mathscr{O}_{X'}) \simeq \bigoplus_{n\geq 0} I^n\mathscr{F}.$$

Since X is Noetherian, in particular quasi-compact and quasi-separated, $\Gamma(X, -)$ commutes with direct sums, so cohomology does as well. We conclude that

$$H^k(X, \pi_*\pi^*\mathscr{F}) \simeq \bigoplus_{n\geq 0} H^k(X, I^n\mathscr{F})$$

is a finitely generated S-module. □

Lemma 1.3.2. *Under the assumptions of 1.3, for every $k \geq 0$, there exists $c \geq 0$ such that for $n \geq c$:*

(i) The multiplication map $I^{n-c} \otimes H^k(X, I^c\mathscr{F}) \to H^k(X, I^n\mathscr{F})$ is surjective.
(ii) For $m \geq 0$, the image of $H^k(X, I^{m+n}\mathscr{F}) \to H^k(X, I^m\mathscr{F})$ is contained in $I^{n-c}H^k(X, I^m\mathscr{F})$.

Proof. (i) From 1.3.1 we know there are finitely many $x_i \in H^k(X, I^{d_i}\mathscr{F})$ for $d_i \geq 0$ that generate $\bigoplus_{n\geq 0} H^k(X, I^n\mathscr{F})$ as an S-module. Setting $c = \max\{d_i\}$ proves (i).

(ii) Let $b = \max(0, m-c)$. We have a commutative diagram

$$\begin{tikzcd}
I^{n+m-c-b} \otimes I^b \otimes H^k(X, I^c\mathscr{F}) \arrow[r, two heads] \arrow[d] & I^{n+m-c} \otimes H^k(X, I^c\mathscr{F}) \arrow[r, two heads] & H^k(X, I^{n+m}\mathscr{F}) \arrow[d] \\
I^{n+m-c-b} \otimes H^k(X, I^m\mathscr{F}) \arrow[rr] & & H^k(X, I^m\mathscr{F})
\end{tikzcd}$$

where surjectivity follows from (i). As $n+m-c-b \geq n-c$, this proves (ii). $\square$

From the discussion in 1.2, we have natural maps $R^k f_*(\mathscr{F}) \to R^k f_*(\mathscr{F}_n)$. Taking global sections gives maps $H^k(X, \mathscr{F}) \to H^k(X, \mathscr{F}_n)$. Similarly, we have maps of the inverse system $H^k(X, \mathscr{F}_{n+m}) \to H^k(X, \mathscr{F}_n)$.

Lemma 1.3.3. *Under the assumptions of 1.3, the following hold for fixed $k \geq 0$.*
(i) There exists $c_1 \geq 0$ such that for $n \geq c_1$,

$$\ker\big(H^k(X, \mathscr{F}) \to H^k(X, \mathscr{F}_n)\big) \subset I^{n-c_1} H^k(X, \mathscr{F}).$$

(ii) For any $n \geq 0$, there exists $c_2(n) \geq n$ such that for $m \geq c_2(n)$,

$$\operatorname{Im}\big(H^k(X, \mathscr{F}_{n+m}) \to H^k(X, \mathscr{F}_n)\big) = \operatorname{Im}\big(H^k(X, \mathscr{F}) \to H^k(X, \mathscr{F}_n)\big)$$

(iii) The inverse system $\big(H^k(X, \mathscr{F}_n)\big)_{n\geq 1}$ satisfies the Mittag-Leffler condition.

Proof. (i) Let $c_1 = \max(c_k, c_{k+1})$ where c_k, c_{k+1} are the constants from 1.3.2. We have a short exact sequence $0 \to I^n\mathscr{F} \to \mathscr{F} \to \mathscr{F}_n \to 0$, which can be checked locally. The long exact sequence of cohomology gives

$$H^k(X, I^n\mathscr{F}) \to H^k(X, \mathscr{F}) \to H^k(X, \mathscr{F}_n)$$

Using 1.3.2(ii), we have

$$\begin{aligned}
\ker\big(H^k(X, \mathscr{F}) \to H^k(X, \mathscr{F}_n)\big) &= \operatorname{Im}\big(H^k(X, I^n\mathscr{F}) \to H^k(X, \mathscr{F})\big) \\
&\subset I^{n-c_1} H^k(X, \mathscr{F})
\end{aligned}$$

for $n \geq c_1$, which proves (i).

(ii) For arbitrary $m \geq 0$, we have a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I^n\mathscr{F} \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}_n \arrow[r] & 0 \\
0 \arrow[r] & I^{n+m}\mathscr{F} \arrow[r] \arrow[u] & \mathscr{F} \arrow[r] \arrow[u, "\mathrm{id}"] & \mathscr{F}_{n+m} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

which gives a map of distinguished triangles. Applying $R\Gamma_X$ and taking cohomologies, we get the commutative diagram

$$\begin{tikzcd}
H^k(X, \mathscr{F}) \arrow[r] & H^k(X, \mathscr{F}_n) \arrow[r, "\partial"] & H^{k+1}(X, I^n\mathscr{F}) \\
H^k(X, \mathscr{F}) \arrow[r] \arrow[u, "\mathrm{id}"] & H^k(X, \mathscr{F}_{n+m}) \arrow[r] \arrow[u] & H^{k+1}(X, I^{n+m}\mathscr{F}) \arrow[u, "\alpha"']
\end{tikzcd}$$

By the Artin-Rees Lemma, there exists $c_3(n) \geq 1$ such that for $N \geq c_3(n)$,

$$\begin{aligned}
I^N H^{k+1}(X, I^n\mathscr{F}) \cap \operatorname{Im}\partial &= I^{N-c_3(n)}\Big(I^{c_3(n)} H^{k+1}(X, I^n\mathscr{F}) \cap \operatorname{Im}\partial\Big) \\
&\subset \partial\Big(I^{N-c_3(n)} H^k(X, \mathscr{F}_n)\Big)
\end{aligned}$$

By 1.3.2(ii), if $m \geq c_1$, we have $\operatorname{Im}\alpha \subset I^{m-c_1}H^{k+1}(X, I^n\mathscr{F})$. Since I^n kills $H^k(X, \mathscr{F}_n)$, we deduce that for $m \geq c_3(n) + c_1 + n$, the intersection $\operatorname{Im}\alpha \cap \operatorname{Im}\partial$ is zero. Thus (ii) holds by setting $c_2(n) = c_3(n) + c_1 + n$.

(iii) follows from (ii) and the definition of the Mittag-Leffler condition. □

We are now ready to prove the theorem on formal functions.

Proof of Theorem 1.2.1. It suffices to check that the map on sections is an isomorphism on the basis of open affines of Y. Since higher direct images are compatible with restriction on the base (cf. [Har77, III.8.2]), we have reduced to proving

$$\widehat{H^k(X, \mathscr{F})} \to \varprojlim_n H^k(X, \mathscr{F}_n)$$

is an isomorphism when $Y = \operatorname{Spec} A$ is affine. Fix $k \geq 0$ and denote $M = H^k(X, \mathscr{F})$ and $M_n = H^k(X, \mathscr{F}_n)$. We want to prove $\widehat{M} \simeq \varprojlim M_n$ where completion is with respect to $\Gamma(\operatorname{Spec} A, \mathscr{I}) =: I \subset A$.

The natural map is the one induced by $M \to M_n$. We define a map in the reverse direction. Take $(x_n) \in \varprojlim M_n$. Since each x_n lies in the image of $M_{n+m} \to M_n$ for $m \geq 0$, 1.3.3(ii) implies that $x_n \in \operatorname{Im}(M \to M_n)$. By 1.3.3(i), $\ker(M \to M_n) \subset I^{n-c_1}M$. Therefore we have

$$M \to \operatorname{Im}(M \to M_n) \to M/I^{n-c_1}M \to M/I^nM$$

for $n \geq c_1$. Let y_n be the image of x_n in M/I^nM. Since $M \to M_{n+1} \to M_n$ commutes, we see that $(y_n) \in \widehat{M}$. It is easy to see that the two maps defined are inverses. Hence $\widehat{M} \simeq \varprojlim M_n$. □

1.4. **Special case.** Let $f : X \to Y$ be a proper map of Noetherian schemes, and $\mathscr{F}$ a coherent sheaf on X. For $y \in Y$ and $n \geq 1$, take the fibered product

$$\begin{tikzcd} X_n \arrow[r, "v_n"] \arrow[d, "f_n"] & X \arrow[d, "f"] \\ \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r, "u_n"] & Y \end{tikzcd}$$

where X_n is a "thickened fiber" of X over y. Let $\mathscr{F}_n = v_n^*\mathscr{F}$ and denote

$$\widehat{R^kf_*(\mathscr{F})}_y = \varprojlim_n R^kf_*(\mathscr{F}) \otimes \mathscr{O}_{Y,y}/\mathfrak{m}_y^n.$$

Lemma 1.4.1. *Let $f : X \to Y$ be a proper map of Noetherian schemes, $\mathscr{F}$ a coherent sheaf on X, and $y \in Y$ a point. There exists a canonical isomorphism*

$$\widehat{R^kf_*(\mathscr{F})}_y \simeq \varprojlim_n H^k(X_n, \mathscr{F}_n)$$

for all $k \geq 0$.

Proof. First, consider the following two fibered product diagrams.

$$\begin{tikzcd} X'_n \arrow[r, "v'_n"] \arrow[d, "f'_n"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r, "u'_n"] & \operatorname{Spec}(\mathscr{O}_{Y,y}) \arrow[r, "g"] & Y \end{tikzcd}$$

where g is flat. By general properties of fibered products, we have $X'_n \simeq X_n$, so $v_n = g' \circ v'_n$, $u_n = g \circ u'_n$, and $f_n = f'_n$. For the left hand side,

$$R^k f_*(\mathscr{F}) \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_{Y,y}/\mathfrak{m}_y^n \simeq g^* R^k f_*(\mathscr{F}) \underset{\mathscr{O}_{Y,y}}{\otimes} \mathscr{O}_{Y,y}/\mathfrak{m}_y^n$$

Since g is flat and f is separated of finite type, by [Har77, III.9.3] or [Gro61, I.4.15], there is a canonical isomorphism $g^* R^k f_*(\mathscr{F}) \simeq R^k f'_*(g'^*\mathscr{F})$. For the right hand side, $X_n \simeq X'_n$ and $\mathscr{F}_n \simeq v'^*_n(g'^*\mathscr{F})$. Therefore we can base change from Y to $\operatorname{Spec} \mathscr{O}_{Y,y}$ to assume Y is a local ring.

Assume $Y = \operatorname{Spec} A$ is local with maximal ideal $\mathfrak{m}$. By 1.2.1, there is a canonical isomorphism of sheaves

$$\widehat{R^k f_*(\mathscr{F})} \simeq \varprojlim_n R^k f_*(v_{n*} v_n^* \mathscr{F}).$$

Taking global sections gives an isomorphism

$$\widehat{H^k(X, \mathscr{F})} \simeq \varprojlim_n H^k(X, v_{n*} v_n^* \mathscr{F}). \tag{2}$$

Since u_n is a closed embedding, so is v_n by base change. Hence $R v_{n*} \simeq v_{n*}$, so by Leray's spectral sequence, $H^k(X, v_{n*} v_n^* \mathscr{F}) \simeq H^k(X_n, \mathscr{F}_n)$. Since A is local, $R^k f_*(\mathscr{F})_{\mathfrak{m}} = \Gamma(Y, R^k f_*(\mathscr{F})) \simeq H^k(X, \mathscr{F})$. Thus (2) is the desired isomorphism, and we are done. □

2. Stein Factorization

2.1. Now that we have proved the theorem on formal functions, we can deduce many important results from it. This section follows [Gro61, Ch. III, 4.3] and [Har77, III.11].

Lemma 2.1.1. *Let $f : X \to Y$ be a proper map of Noetherian schemes such that $\mathscr{O}_Y \simeq f_* \mathscr{O}_X$. Then the fibers $f^{-1}(y)$ are connected and nonempty for all $y \in Y$.*

Proof. The isomorphism implies f is dominant. Since f is proper, it is closed and therefore surjective on points. Hence all fibers are nonempty. Fix $y \in Y$. Let $f_n : X_n \to \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^n)$ be as in 1.4. For $n \geq 1$, all X_n have the same underlying space $f^{-1}(y)$. Suppose $f^{-1}(y) = X' \sqcup X''$ is disconnected. Then for each n, we have a direct sum of nonzero rings

$$H^0(X_n, \mathscr{O}_{X_n}) = H^0(X', \mathscr{O}_{X_n}) \oplus H^0(X'', \mathscr{O}_{X_n}).$$

By 1.4.1, taking $\mathscr{F} = \mathscr{O}_X$ we have

$$\widehat{\mathscr{O}_{Y,y}} \simeq (\widehat{f_* \mathscr{O}_X})_y \simeq \varprojlim_n H^0(X_n, \mathscr{O}_{X_n}) \simeq \varprojlim_n H^0(X', \mathscr{O}_{X_n}) \oplus \varprojlim_n H^0(X'', \mathscr{O}_{X_n})$$

as rings. Each inverse limit is nonzero, since the unit 1 is distinct from 0. However a local ring is never a direct sum of nonzero rings, so we have a contradiction. □

Theorem 2.1.2 (Zariski's Connectedness Theorem)**.** *Let $f : X \to Y$ be a proper map of Noetherian schemes. Then f can be factored into $X \xrightarrow{f'} Y' \xrightarrow{g} Y$ where $\mathscr{O}_{Y'} \simeq f'_* \mathscr{O}_X$, f' is proper with connected nonempty fibers, and g is finite.*

The factorization of f into $g \circ f'$ is called the *Stein factorization.*

Proof. Since f is proper, $f_*\mathscr{O}_X$ is a coherent $\mathscr{O}_Y$-algebra. Let $Y' = \mathbf{Spec}_Y(f_*\mathscr{O}_X)$ be the relative Spec. Then $g : Y' \to Y$ is affine by construction, and finite since $f_*\mathscr{O}_X$ is coherent. Let $\mathscr{A} = f_*\mathscr{O}_X$. Recall that

$$\mathrm{Hom}(S, \mathbf{Spec}_Y(\mathscr{A})) = \{\phi : S \to Y, \mathscr{A} \to \phi_*\mathscr{O}_S\}$$

Thus we have a map $f' : X \to Y'$ corresponding to $f : X \to Y$ and $\mathscr{A} \xrightarrow{\sim} f_*\mathscr{O}_X$. Then

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes. Finite maps are proper, so g is proper. This implies f' is also proper. By construction, $\mathscr{O}_{Y'} \to f'_*\mathscr{O}_X$ is an isomorphism, corresponding to $\mathscr{A} \simeq f_*\mathscr{O}_X$. Hence f' has connected nonempty fibers by 2.1.1. □

Corollary 2.1.3. *Under the assumptions of 2.1.2, for all $y \in Y$, the connected components of $f^{-1}(y)$ are in bijection with the finite collection of points in the fiber $g^{-1}(y)$.* □

Remark 2.1.4. Note that if X is irreducible in 2.1.2, then $f'(X) = Y'$ is irreducible. Since $Y' = \mathbf{Spec}_Y(f_*\mathscr{O}_X)$, if X is reduced, then so is Y'.

Lemma 2.1.5. *Let $g : Y' \to Y$ be a birational finite map between Noetherian integral schemes. In addition, assume Y is normal. Then g is an isomorphism.*

Proof. The question is local, so assume $Y = \operatorname{Spec} A$ is affine. Then $Y' = \operatorname{Spec} B$ is also affine. By birationality, $\mathrm{Frac}(B) = \mathrm{Frac}(A)$. Since B is finite as an A module, by normality $A = B$. □

Corollary 2.1.6. *Let $f : X \to Y$ be a proper birational map of Noetherian integral schemes. In addition, assume Y is normal. Then the Stein factorization of f is trivial. In particular, f has connected fibers.*

Proof. Consider the Stein factorization $X \xrightarrow{f'} Y' \xrightarrow{g} Y$. By 2.1.4, Y' is integral. Since f is birational, g is also birational. Now 2.1.5 implies $Y' \simeq Y$. The rest follows from 2.1.2. □

3. Zariski's Main Theorem

3.1. This section follows [Gro61, Ch. III, 4.4].

Remark 3.1.1. Let $X \to \operatorname{Spec} k$ be a scheme of finite type over a field k. Then an isolated, i.e., open, point $x \in X$ is also closed (and hence a connected component). For any affine $x \in \operatorname{Spec} A$, there is $f \in A$ such that $\operatorname{Spec} A_f = \{x\}$. Since A is a k-algebra of finite type, $\operatorname{Spec} A_f \to \operatorname{Spec} A$ sends closed points to closed points. Therefore x is closed in $\operatorname{Spec} A$. This is true for any affine, so x is closed in X.

Lemma 3.1.2. *Suppose $f : X \to Y$ is a proper map of Noetherian schemes. Let X' be the set of points $x \in X$ that are isolated in the fiber $f^{-1}(f(x))$. Then $X' \subset X$ is open, and if $f = g \circ f'$ is the Stein factorization of f, we have $f'(X') \subset Y'$ is open, and $f'|_{X'} : X' \to f'(X')$ is an isomorphism.*

Proof. From 2.1.3 we see that x is isolated in $f^{-1}(f(x))$ if and only if it is isolated in $f'^{-1}(f'(x))$. Therefore we can assume $f = f'$, so $\mathscr{O}_Y \simeq f_*\mathscr{O}_X$. Fix $x \in X'$ and set $y = f(x)$. By 2.1.2, $f^{-1}(y)$ is connected. Since $X_y \to \operatorname{Spec} k_y$ is of finite type, 3.1.1 implies x is a connected component of $f^{-1}(y)$; hence $f^{-1}(y) = \{x\}$. Take open affines $y \in \operatorname{Spec} A \subset Y$ and $x \in \operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$. Since f is proper, hence closed, $U = Y - f(X - \operatorname{Spec} B)$ is an open neighborhood of y such that $f^{-1}(U) \subset \operatorname{Spec} B$. Therefore there exists a basic open $\operatorname{Spec} A_f \subset U \cap \operatorname{Spec} A$ such that $f^{-1}(\operatorname{Spec} A_f) = (f|_{\operatorname{Spec} B})^{-1}(\operatorname{Spec} A_f) = \operatorname{Spec} B_f$. Now the isomorphism $\mathscr{O}_Y \simeq f_*\mathscr{O}_X$ restricted to $\operatorname{Spec} A_f$ gives $A_f \simeq B_f$. Thus

$$f^{-1}(\operatorname{Spec} A_f) = \operatorname{Spec} B_f \xrightarrow{\sim} \operatorname{Spec} A_f,$$

which also implies $\operatorname{Spec} B_f \subset X'$. Gluing for all $x \in X'$ proves the claim. □

Remark 3.1.3. Let $X \to \operatorname{Spec} k$ be a scheme of finite type over a field k such that X has finitely many points. For any $\operatorname{Spec} A \subset X$, there are finitely many maximal ideals $\mathfrak{m}_i \subset A$. We have $\bigcup V(\mathfrak{m}_i) = V(\bigcap \mathfrak{m}_i) = V(\operatorname{rad}(0)) = \operatorname{Spec} A$ (here we use A is of finite type over k). Therefore X is topologically discrete.

Lemma 3.1.4. *Let $f : X \to Y$ be a map of Noetherian schemes. TFAE:*
(i) f is finite.
(ii) f is affine and proper.
(iii) f is proper and for all $y \in Y$, $f^{-1}(y)$ is a finite set.

Proof. We already know (i) implies (ii). If f is affine and proper, then by base change, so is $f' : X_y \to \operatorname{Spec} k_y$. Hence $X_y = \operatorname{Spec} A$ and $f'_*\mathscr{O}_{X_y}$ is coherent on k_y. Therefore A is a finite k_y-vector space, hence Artinian. Thus (ii) implies (iii). Lastly, assume f is proper and quasi-finite. Then X_y is of finite type over $\operatorname{Spec} k_y$. By 3.1.3, X_y is discrete. Using the notation of 3.1.2, we deduce that $X = X'$. Therefore $f' : X \to Y'$ is an isomorphism. Since g is finite, this proves (iii) implies (i). □

A map of schemes is called *compactifiable* if it factors as an open embedding followed by a proper map. In particular, quasi-projective maps are compactifiable.

Theorem 3.1.5 (Zariski's Main Theorem)**.** *Let $f : X \to Y$ be a compactifiable map of Noetherian schemes, X' the set of points $x \in X$ that are isolated in the fiber $f^{-1}(f(x))$. Then $X' \subset X$ is an open subset, and the induced subscheme is isomorphic to an open subscheme of a scheme Y' finite over Y.*

Proof. By definition of compactifiable maps, we can factor f as $X \to Z \to Y$ where $X \to Z$ is an open embedding, and $Z \to Y$ is proper. The theorem follows from 3.1.2. □

Remark 3.1.6. If X is reduced (resp. irreducible), then we can assume Y' is reduced (resp. irreducible) in 3.1.6. To see this, we can factor the open embedding $X \to Z$ further as $X \to \overline{X} \to Z$ where $\overline{X}$ is the scheme-theoretic closure in Z (which exists since open embeddings to a Noetherian scheme are quasi-compact and separated). Here $X \hookrightarrow \overline{X}$ is an open embedding and $\overline{X} \hookrightarrow Z$ is a closed embedding (hence proper). Therefore $\overline{X} \to Y$ is proper. If X is reduced, then $\overline{X}$ has the reduced induced structure. If X is irreducible, the closure is also irreducible. Hence Y' is reduced (resp. irreducible) by 2.1.4.

Corollary 3.1.7. *Let Y be a Noetherian scheme, $f : X \to Y$ a map of finite type, and $x \in X$ a point isolated in $f^{-1}(f(x))$. Then there exists an open neighborhood of x that is isomorphic to an open subscheme of a scheme finite over Y.*

Proof. For such a point $x \in X$, set $y = f(x)$. Take open affines $y \in \operatorname{Spec} A \subset Y$ and $x \in \operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$. Then B is a finitely generated A-algebra, so there is a closed embedding $\operatorname{Spec} B \hookrightarrow \mathbf{A}^n_A$ for some n. Then $\operatorname{Spec} B \hookrightarrow \mathbf{A}^n_A \hookrightarrow \mathbf{A}^n_Y$ is a closed embedding followed by an open embedding. Since all schemes are Noetherian, open embeddings are quasi-compact and separated. Thus a closed embedding followed by an open embedding is equivalent to a locally closed embedding, i.e., an open embedding followed by a closed embedding. In particular, compositions of locally closed embeddings are still locally closed embeddings. Hence we have $\operatorname{Spec} B \to \mathbf{A}^n_Y \to \mathbf{P}^n_Y$ a locally closed embedding over Y. In other words, $\operatorname{Spec} B \to Y$ is quasi-projective. The claim now follows from 3.1.5. $\square$

Corollary 3.1.8. *Let $f : X \to Y$ be a birational map of finite type between integral Noetherian schemes. In addition, assume Y is normal and the fibers $f^{-1}(y)$ are finite for all $y \in Y$. Then f is an open embedding; if moreover f is closed (e.g., if f is proper), then f is an isomorphism.*

Proof. For $x \in X$, let $y = f(x)$. Then by 3.1.3, $f^{-1}(y)$ is discrete. From the proof of 3.1.7, there is affine $x \in \operatorname{Spec} B$ such that $\operatorname{Spec} B \to Y$ is quasi-projective. By 3.1.6, this map factors as $\operatorname{Spec} B \to Z \to Y$, where $\operatorname{Spec} B \hookrightarrow Z$ is an open embedding, $Z \to Y$ is projective, and Z is integral. Since f is birational, so is $Z \to Y$. Then 2.1.6 implies the Stein factorization of $Z \to Y$ is trivial. Now 3.1.5 implies there is a neighborhood $x \in U$ in X such that $f|_U : U \to f(U)$ is an isomorphism onto an open subset of Y. This is true for all $x \in X$, so by gluing we deduce that f is an open embedding. Since f is dominant, if it is also closed we have $f(X) = Y$, so f is an isomorphism. $\square$

3.2. In the language of commutative algebra, 3.1.7 translates to:

Corollary 3.2.1. *Let A be a Noetherian ring, B a finitely generated A-algebra, $\mathfrak{p} \subset B$ a prime, and $\mathfrak{q} \subset A$ the contraction. Suppose that $\mathfrak{p}$ is both a maximal and minimal prime lying over $\mathfrak{q}$. Then there exists $g \in B - \mathfrak{p}$, an A-algebra A' that is finite as an A-module, and $f' \in A'$ such that B_g and $A'_{f'}$ are isomorphic as A-algebras.* $\square$

There is a more specific version of 3.2.1:

Corollary 3.2.2. *Let A be a Noetherian local ring, B a finitely generated A-algebra, $\mathfrak{p} \subset B$ a maximal ideal lying over the maximal ideal $\mathfrak{m} \subset A$. Suppose $\mathfrak{p}$ is also minimal among primes over $\mathfrak{m}$. Then there exists an A-algebra A' that is finite as an A-module and a maximal ideal $\mathfrak{m}' \subset A'$ (which contracts to $\mathfrak{m}$) such that $B_{\mathfrak{p}}$ is isomorphic to $A'_{\mathfrak{m}'}$ as A-algebras.*

Proof. From 3.2.1, we have $B_g \simeq A'_{f'}$ over A, where $g \in B - \mathfrak{p}$. Thus $\mathfrak{p} \in \operatorname{Spec} B_g$ corresponds to $\mathfrak{m}' \in \operatorname{Spec} A'_{f'}$, which we consider as a prime in A'. Since $\mathfrak{p}$ lies over $\mathfrak{m}$, we have $\mathfrak{m}'$ lying over $\mathfrak{m}$. Since A' is a finite A-module, $\operatorname{Spec}(A' \otimes k_{\mathfrak{m}})$ is Artinian. Therefore $\mathfrak{m}'$ is a maximal prime lying over $\mathfrak{m}$, and hence a maximal ideal in A'. Localizing, we have $B_{\mathfrak{p}} \simeq A'_{\mathfrak{m}'}$. $\square$

Corollary 3.2.3. *Under the conditions of 3.2.2, suppose in addition that A, B are integral with the same field of fractions K. If A is normal, then $A = B$.*

Proof. From 3.1.6, we can assume A' is integral, and the isomorphism $B_{\mathfrak{p}} \simeq A'_{\mathfrak{m}'}$ implies $\mathrm{Frac}(A') = K$. Since A is normal and A' is a finite A-module, we have $A = A'$. Thus $B_{\mathfrak{p}} = A$. We also have $A \subset B \subset B_{\mathfrak{p}}$, which implies $A = B$. □

4. Grothendieck's Generalization

4.1. Grothendieck's generalization of Zariski's Main Theorem deals with quasi-finite maps of schemes.

Corollary 4.1.1. *Let $f : X \to Y$ be a compactifiable map of Noetherian schemes. If f is quasi-finite, then f can be factored as $X \xrightarrow{j} Y' \xrightarrow{g} Y$ where j is an open embedding and g is finite.*

Proof. This follows from 3.1.3 and 3.1.5. □

Grothendieck generalized this result by removing some of the conditions. His proof uses deep commutative algebra but no cohomology.

Theorem 4.1.2 ([Gro66, Ch. IV, 8.12])**.** *Let Y be a quasi-compact, quasi-separated scheme. If a map $f : X \to Y$ is quasi-finite, separated, and of finite presentation, then f can be factored into an open embedding followed by a finite map.*

Alternatively and extremely overkill, Nagata's compactification theorem [Con] says any separated map of finite type between quasi-compact, quasi-separated schemes can be factored into an open embedding followed by a proper map, i.e., is compactifiable. Using this theorem, 4.1.1 implies Grothendieck's result in the Noetherian case.

References

[Con] Brian Conrad. Deligne's notes on Nagata compactifications. `http://math.stanford.edu/~conrad/papers/nagatafinal.pdf`.

[Gro61] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[Gro66] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[sta] *Stacks Project.* `http://math.columbia.edu/algebraic_geometry/stacks-git`.