RADON INVERSION FORMULAS OVER LOCAL FIELDS

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Abstract. Let F be a local field and $n \geq 2$ an integer. We study the Radon transform as an operator $M : \mathcal{C}_+ \to \mathcal{C}_-$ from the space of smooth K-finite functions on $F^n \setminus \{0\}$ with bounded support to the space of smooth K-finite functions on $F^n \setminus \{0\}$ supported away from a neighborhood of 0. These spaces naturally arise in the theory of automorphic forms. We prove that M is an isomorphism and provide formulas for M^{-1}. In the real case, we show that when K-finiteness is dropped from the definitions, the analog of M is not surjective.

1. Introduction

1.1. Some notation.

1.1.1. Let F be a local field (i.e., F is either non-Archimedean or $\mathbb{R}$ or $\mathbb{C}$). Let G denote the topological group $\mathrm{GL}_n(F)$ for an integer $n \geq 2$.

Let K be the standard maximal compact subgroup of G (i.e., if F is non-Archimedean then $K = \mathrm{GL}_n(\mathcal{O})$, where $\mathcal{O} \subset F$ is the ring of integers, if $F = \mathbb{R}$ then $K = O(n)$, and if $F = \mathbb{C}$ then $K = U(n)$).

1.1.2. We fix a field E of characteristic 0; if F is Archimedean we assume that E equals $\mathbb{C}$. Unless otherwise specified, all functions will take values in E.

1.1.3. Let $\mathcal{C}$ denote the space of K-finite C^∞ functions on $F^n \setminus \{0\}$. In §2.2 we define the subspace $\mathcal{C}_+ \subset \mathcal{C}$ consisting of functions with bounded support and the subspace $\mathcal{C}_- \subset \mathcal{C}$ consisting of functions supported away from a neighborhood of 0.

1.2. **Subject of this article.** In this article we consider the Radon transform as an operator

$$M : \mathcal{C}_+ \to \mathcal{C}_-.$$

When F is non-Archimedean, M is known to be an isomorphism [BK]. An explicit formula for the inverse was, however, not present in the literature. There is a 'classical' inversion formula due to Černov [Ch] on the space of Schwartz functions, but its relation to M^{-1} is not obvious. We formulate and prove a simple formula for M^{-1} in the non-Archimedean case and relate it to Černov's formula.

In the Archimedean case, the invertibility of M was *a priori* unclear due to the nonstandard nature of the function spaces $\mathcal{C}_\pm$. We prove that M is indeed an isomorphism when F is Archimedean and provide formulas for M^{-1} (here K-finiteness of $\mathcal{C}$ plays a crucial role).

1.3. **Motivation.** Our interest in the operator M originates from the classical theory of automorphic forms. Let G denote the algebraic group SL_2 and N (resp. N^-) the subgroup of strictly upper (resp. lower) triangular matrices and T the maximal torus of diagonal matrices. Then $G(F)/N(F) = F^2 \setminus \{0\}$ and $G(F)/N^-(F) = F^2 \setminus \{0\}$. The operator M is the standard (local) intertwiner $M : \mathcal{C}_+(G(F)/N(F)) \to \mathcal{C}_-(G(F)/N^-(F))$.

While we work only with the local field F, one gets a global analog of M by considering the standard intertwiner $M : \mathcal{C}_+(G(\mathbb{A})/T(F)N(\mathbb{A})) \to \mathcal{C}_-(G(\mathbb{A})/T(F)N(\mathbb{A}))$ where F is a global field and $\mathbb{A}$ the adele ring. The intertwiner plays an important role in the theory of Eisenstein series and their constant terms [Bu, §3.7]. The constant terms of automorphic forms reside in the space $\mathcal{C}_-(G(\mathbb{A})/T(F)N(\mathbb{A}))$, which makes it a natural space to study in this setting. The results of this article are used to prove invertibility of the global intertwiner in [DW].

In the situation where F is a non-Archimedean local field, the operator M^{-1} is essentially the same as the 'Bernstein map' introduced in [BK, Definition 5.3]; the precise relation between the two is explained in [BK, Theorem 7.5]. The Bernstein map is also studied in [SV] (there it is called the asymptotic map) in the more general context of spherical varieties.

In the real case, the Radon transform has been studied extensively by analysts ([H1], [H2], [H3]) over slightly different function spaces.

1.4. **Structure of the article.** In §2 we define the subspaces $\mathcal{C}_\pm \subset \mathcal{C}$ and recall the definition of the Radon transform over a general local field F.

In §3, we consider the case when F is non-Archimedean. We prove that M is invertible and give a formula for M^{-1} in Theorem 3.2.6. This is done by relating the Radon transform to the Fourier transform (§3.3-3.5). We deduce the previously known Radon inversion formula of Černov [Ch] from Theorem 3.2.6 in §3.6.

We consider the real case in §4. The formula for M^{-1} is given on each K-isotypic component of $\mathcal{C}_-$ in Theorem 4.3.3 in terms of convolution with a distribution on $\mathbb{R}_{>0}$. The Mellin transform of this distribution is computed in Theorem 4.4.1. The proof of the theorems is in §4.6. The invertibility of M heavily relies on the K-finiteness assumption in the definition of $\mathcal{C}$. In §4.7, we prove (Corollary 4.7.4) that the analog of M is not surjective when K-finiteness is dropped from the definitions.

In §5, the complex case is developed in the same way as the real case. The inversion formula is given in Theorem 5.3.3 and the reformulation using the Mellin transform is Theorem 5.4.1.

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2. Recollections on the Radon transform

2.1. **The norm on F^n.** Let $|\cdot|$ denote the normalized absolute value on F when F is non-Archimedean and the usual absolute value[1] when F is Archimedean. For $a \in F^\times$, set $v(a) := -\log|a|$. If F is non-Archimedean log stands for $\log_q$, where q is the order of the residue field of F. If F is Archimedean, log is understood as the natural logarithm.

We define a norm $\|\cdot\|$ on F^n as follows. If F is non-Archimedean, then $\|\cdot\|$ is the norm induced by the standard lattice $\mathcal{O}^n$ (i.e., $\|x\|$ is the maximum of the absolute values of the coordinates of $x \in F^n$). If F is Archimedean, then $\|\cdot\|$ is induced by the standard Euclidean/Hermitian inner product (i.e., the square root of the sum of the absolute values squared).

For $x \in F^n \setminus \{0\}$, set $v(x) := -\log\|x\|$.

2.2. **The spaces $\mathcal{C}, \mathcal{C}_c, \mathcal{C}_\pm$.** Let $\mathcal{C}$ denote the space of K-finite C^∞ functions on $F^n \setminus \{0\}$ (recall that if F is non-Archimedean, C^∞ means locally constant). Let $\mathcal{C}_c \subset \mathcal{C}$ be the subspace of compactly supported functions on $F^n \setminus \{0\}$.

Given a real number R, let $\mathcal{C}_{\le R} \subset \mathcal{C}$ denote the set of all functions $\varphi \in \mathcal{C}$ such that $\varphi(\xi) \neq 0$ only if $v(\xi) \le R$. Similarly, we have $\mathcal{C}_{\ge R}, \mathcal{C}_{>R}$, and so on. Let $\mathcal{C}_-$ denote the union of the

[1] If $F = \mathbb{R}$, then the normalized absolute value coincides with the usual absolute value. If $F = \mathbb{C}$, then the normalized absolute value is the square of the usual absolute value.

subspaces $\mathcal{C}_{\leq R}$ for all R. Let $\mathcal{C}_+$ denote the union of the subspaces $\mathcal{C}_{\geq R}$ for all R. Clearly $\mathcal{C}_- \cap \mathcal{C}_+ = \mathcal{C}_c$ and $\mathcal{C}_- + \mathcal{C}_+ = \mathcal{C}$.

2.3. **Radon transform.** Equip F with the following Haar measure: if F is non-Archimedean we require that $\mathrm{mes}(\mathcal{O}) = 1$; if F is Archimedean we use the usual Lebesgue measure. Let the measure on F^n be the product of the measures on n copies of F. Fix the Haar measure on $F^\times$ to be $d^\times t := \frac{dt}{|t|}$.

Let $f \in \mathcal{C}_+$. The Radon transform $\mathcal{R}f(\xi, s)$, for $\xi \in F^n \setminus \{0\}$ and $t \in F^\times$, is defined by the formula

$$\mathcal{R}f(\xi, t) = \int_{F^n} f(x)\delta(\xi \cdot x - t)dx,$$

where $\xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$ and δ is the delta distribution on F. The expression for $\mathcal{R}f(\xi, t)$ can also be written directly as

$$\mathcal{R}f(\xi, t) = \int_{\xi \cdot x = t} f(x) d\mu_\xi$$

where $d\mu_\xi$ is the measure on the hyperplane $\xi \cdot x = t$ such that $d\mu_\xi dt = dx$. We get an operator $M : \mathcal{C}_+ \to \mathcal{C}_-$ by setting

$$Mf(\xi) = \int_{\xi \cdot x = 1} f(x) d\mu_\xi. \tag{2.1}$$

Proposition 2.3.1. *For any number R one has $M(\mathcal{C}_{\geq R}) \subset \mathcal{C}_{\leq -R}$.*

Proof. Let $f \in \mathcal{C}_{\geq R}$ and $\xi \in F^n \setminus \{0\}$ with $v(\xi) > -R$. Then $\xi \cdot x = 1$ implies $v(x) < R$, so $f(x) = 0$. Therefore $Mf \in \mathcal{C}_{\leq -R}$. □

2.3.2. The natural action of G on $F^n \setminus \{0\}$ induces a G-action on $\mathcal{C}$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $g \in G, f \in \mathcal{C}, x \in F^n \setminus \{0\}$. Then

$$M(g \cdot f) = |\det g|^d (g^T)^{-1} Mf \tag{2.2}$$

for $f \in \mathcal{C}_+$ and $g \in G$, where g^T is the transpose matrix, and $d = 1$ if $F \neq \mathbb{C}$ and $d = 2$ if $F = \mathbb{C}$ (i.e., $|\det g|^d$ is the normalized absolute value of $\det g$).

3. F non-Archimedean

In this section we consider the case when F is a non-Archimedean local field. Let $\mathcal{O}$ the ring of integers, $\mathfrak{p}$ the maximal ideal, ϖ a uniformizer, and $\mathbb{F}_q$ the residue field of F.

The main result of this section is Theorem 3.2.6. In order to state the theorem, we must first define a new operator $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$, which is done in §3.2.

3.1. **K-finite functions.** The action of G on $F^n \setminus \{0\}$ is continuous and transitive. Since F is non-Archimedean, K is an open subgroup of G, and we have the following description of K-finite functions.

Lemma 3.1.1. *A C^∞ function φ on $F^n \setminus \{0\}$ is K-finite if and only if there exists an open subgroup $H \subset K$ such that $\varphi(h\xi) = \varphi(\xi)$ for all $h \in H$ and $\xi \in F^n \setminus \{0\}$.*

Proof. Let W denote the span of the K translates of f. By assumption W is finite dimensional, and this implies that there exists a compact open subset X of $F^n \setminus \{0\}$ such that the restriction map $W \to C^\infty(X)$ is injective. Any locally constant function on X is fixed by an open subgroup of K, which proves the lemma. □

One may sometimes wish to consider the group $\mathrm{SL}_n(F)$ rather than $\mathrm{GL}_n(F)$ acting on $F^n \setminus \{0\}$. The next lemma shows that this does not change the corresponding subspaces of invariant functions in $\mathcal{C}$.

Lemma 3.1.2. *For an integer $r > 0$, set $K_r := \ker(\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{p}^r))$. Then the following properties of a function φ on $F^n \setminus \{0\}$ are equivalent for $n \geq 2$:*
(i) φ is stabilized by $K_r \cap \mathrm{SL}_n(F)$,
(ii) $\varphi(\xi') = \varphi(\xi)$ for $\xi, \xi' \in F^n \setminus \{0\}$ satisfying $v(\xi' - \xi) \geq v(\xi) + r$,
(iii) φ is stabilized by K_r.

Proof. Suppose that φ is stabilized by $K_r \cap G$. Take $\xi, \xi' \in F^n \setminus \{0\}$ with $v(\xi' - \xi) \geq v(\xi) + r$. We can find a basis $v_1, \ldots, v_n$ of $\mathcal{O}^n$ with $v_1 = \varpi^{-v(\xi)}\xi$ and $v_2 = \varpi^{-v(\xi'-\xi)}(\xi' - \xi)$. Let g send v_1 to $v_1 + \varpi^{v(\xi'-\xi)-v(\xi)}v_2$ and v_k to v_k for $k > 1$. Then $g \in K_r \cap G$ and $g\xi = \xi'$. Thus $\varphi(\xi') = \varphi(\xi)$. This proves (i) implies (ii). The other implications are easy. □

3.2. **The operator $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$.** Let $\mathcal{S}'_b(F)$ denote the space of distributions β on F such that for any open subgroup $U \subset \mathcal{O}^\times$, the multiplicative U-average[2] β_U has compact support and $\langle \beta_U, 1 \rangle = 0$. Note that if $\langle \beta_U, 1 \rangle = 0$ for some U, then it is true for all U.

3.2.1. We would like to define $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$ for $\beta \in \mathcal{S}'_b(F)$ by

$$(A_\beta \varphi)(x) = \int_{F^n} \beta(\xi \cdot x)\varphi(\xi) d\xi$$

but we must explain the meaning of the r.h.s.

Fix $\varphi \in \mathcal{C}_-$ and $x \in F^n \setminus \{0\}$. For any open compact subgroup $\Lambda \subset F^n$ let

$$I(\Lambda) := \int_\Lambda \beta(\xi \cdot x)\varphi(\xi) d\xi.$$

Lemma 3.2.2. *There exists Λ such that $I(\Lambda') = I(\Lambda)$ for any Λ' containing Λ.*

Proof. Choose $\xi_0 \in F^n$ such that $\xi_0 \cdot x = 1$ and $v(\xi_0) = -v(x)$. Then $F^n = F\xi_0 \oplus H$ where H is the hyperplane $\{\xi \mid \xi \cdot x = 0\}$. Lemma 3.1.1 implies that $\varphi \in \mathcal{C}_-$ is fixed by the homothety actions of an open subgroup $U \subset \mathcal{O}^\times$. Therefore we can replace β by the multiplicative average β_U. Let $\mathfrak{p}^i \subset F$ be a fractional ideal containing the support of β_U. Lemma 3.1.2(ii) implies that $\varphi(s\xi_0 + \xi) = \varphi(\xi)$ if $s \in \mathfrak{p}^i$ and $v(\xi) \leq v(\xi_0) + i - r$, where φ is stabilized by the congruence subgroup K_r. Put $a := r - i$. Let $\Lambda := \mathfrak{p}^i\xi_0 \oplus \{\xi \in H \mid v(\xi) \geq -v(x) - a\}$.

Now suppose Λ' is a subgroup containing Λ. Define $\Lambda'' = \{\xi \in \Lambda' \mid \xi \cdot x \in \mathfrak{p}^i\} \supset \Lambda$. Then $I(\Lambda') = I(\Lambda'')$ since $\mathfrak{p}^i$ contains the support of β. Now $\Lambda'' = \mathfrak{p}^i\xi_0 \oplus (\Lambda'' \cap H)$. Thus

$$I(\Lambda'') - I(\Lambda) = \int_{\xi \in (\Lambda'' \setminus \Lambda) \cap H} \int_{\mathfrak{p}^i} \beta_U(s)\varphi(s\xi_0 + \xi)|\xi_0| ds d\mu_x.$$

Note that $\xi \in (\Lambda'' \setminus \Lambda) \cap H$ satisfies $v(\xi) < -v(x) - a$ and hence $\varphi(s\xi_0 + \xi) = \varphi(\xi)$. We conclude that $I(\Lambda'') = I(\Lambda)$ since $\langle \beta_U, 1 \rangle = 0$. □

3.2.3. Put $(A_\beta \varphi)(x) := I(\Lambda)$ where Λ is as in Lemma 3.2.2.

Corollary 3.2.4. *Let R be any number. If $\varphi \in \mathcal{C}_{\leq -R}$, then $A_\beta \varphi \in \mathcal{C}_{\geq R-a}$, where a is an integer depending only on β and the stabilizer of φ in G.*

[2]The multiplicative U-average β_U is defined by $\beta_U(t) = \frac{1}{\mathrm{mes}(U)} \int_U \beta(ut) d^\times u$.

Proof. We use the notation from the proof of Lemma 3.2.2. Note that the choice of a is independent of $x \in F^n \setminus \{0\}$. It follows from our definition above and the proof of Lemma 3.2.2 that

$$(A_\beta \varphi)(x) = \int_{\substack{\xi \in H \\ v(\xi) \geq -v(x)-a}} \int_{\mathfrak{p}^i} \beta_U(t)\varphi(s\xi_0 + \xi)|\xi_0| ds d\mu_x,$$

which is zero if $v(x) < R - a$. $\square$

Thus we have defined an operator $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$.

Remark 3.2.5. For $\varphi \in \mathcal{C}_-$ we have $A_\beta(g\varphi) = |\det g|(g^T)^{-1}(A_\beta\varphi)$ where g^T is the transpose. In other words, the operator $\mathcal{C}_- \to \{\text{measures on } (F^n)^* \setminus \{0\}\}$ defined by $\varphi \mapsto (A_\beta\varphi)dx$ is equivariant with respect to the action of G.

The goal of this section is to prove the following.

Theorem 3.2.6. *The operator $M : \mathcal{C}_+ \to \mathcal{C}_-$ is an isomorphism. The inverse of M is A_β, where β is the compactly supported distribution on F equal to*

$$\frac{1 - q^{n-1}}{1 - q^{-n}}(|s-1|^{-n} - |s|^{-n}).$$

The distributions $|s-1|^{-n}$ and $|s|^{-n}$ are defined as in [GGP, Ch. 2, §2.3], i.e.,

$$\langle |s|^{-n}, f\rangle = \int_F |s|^{-n}(f(s) - f(0))ds$$

for a test function $f \in C_c^\infty(F)$.

We prove Theorem 3.2.6 in §3.5.

Remark 3.2.7. Let β be as defined in Theorem 3.2.6. Then the integral of β along any compact open subset of F has value in $\mathbb{Z}[\frac{1}{q}]$. This is not true for the distribution $\frac{1-q^{n-1}}{1-q^{-n}}|s|^{-n}$.

3.3. **Fourier transform.** We assume without loss of generality that E contains all roots of unity. Choose a nontrivial additive character ψ of F which is trivial on $\mathcal{O}$ but nontrivial on $\varpi^{-1}\mathcal{O}$. The Haar measure we chose for F is self-dual with respect to ψ. Note that $\psi \in \mathcal{S}_b'(F)$. Define the Fourier transform $\mathcal{F} : \mathcal{C}_- \to \mathcal{C}_+$ by

$$\mathcal{F} := A_\psi.$$

On the other hand, we also have an operator $\mathcal{F}' : \mathcal{C}_+ \to \mathcal{C}_-$ defined by

$$\mathcal{F}'f(\xi) = \int_{F^n} f(x)(\psi(-\xi \cdot x) - 1)dx. \tag{3.1}$$

Moreover for any number R, one observes that $\mathcal{F}'(\mathcal{C}_{\geq R}) \subset \mathcal{C}_{<-R}$.

Proposition 3.3.1. *The operators $\mathcal{F}$ and $\mathcal{F}'$ are mutually inverse.*

Proof. Proposition 2.3.1 and Corollary 3.2.4 imply that $\mathcal{F}\mathcal{F}'(\mathcal{C}_{\geq R}) \subset \mathcal{C}_{\geq R+a}$ and $\mathcal{F}'\mathcal{F}(\mathcal{C}_{\leq -R}) \subset \mathcal{C}_{\leq -R-a}$ on functions stabilized by K_r for a fixed $r > 0$. As a consequence, it is enough to check the equalities $\mathcal{F}\mathcal{F}' = \mathrm{id}$ and $\mathcal{F}'\mathcal{F} = \mathrm{id}$ on the subspace $\mathcal{C}_c = \mathcal{C}_+ \cap \mathcal{C}_-$.

Let $f \in \mathcal{C}_c$. Then the usual Fourier transform $\hat{f}$ is a compactly supported function on F^n. Note that $\mathcal{F}'f(\xi) = \hat{f}(\xi) - \hat{f}(0)$. By the definition of $\mathcal{F}$, we have

$$\mathcal{F}\mathcal{F}'f(x) = \int_\Lambda (\hat{f}(\xi) - \hat{f}(0))\psi(\xi \cdot x)d\xi$$

for any sufficiently large open compact subgroup $\Lambda \subset F^n$. Since $\hat{f}$ is compactly supported, the usual Fourier inversion formula implies that $\int_\Lambda \hat{f}(\xi)\psi(\xi \cdot x)d\xi = f(x)$ if Λ contains the support of $\hat{f}$. Since x is nonzero, $\int_\Lambda \psi(\xi \cdot x)d\xi = 0$ for Λ large enough. Therefore $\mathcal{F}\mathcal{F}'f = f$.

In the other direction, let $\varphi \in \mathcal{C}_c$. Then $\mathcal{F}\varphi(x) = \hat{\varphi}(-x)$ is compactly supported on F^n. Again the Fourier inversion formula implies that

$$\mathcal{F}'\mathcal{F}\varphi(\xi) = \int_{F^n} \hat{\varphi}(x)\psi(-\xi \cdot x)dx - \int_{F^n} \hat{\varphi}(x)dx = \varphi(\xi) - \varphi(0) = \varphi(\xi). \qquad \square$$

3.4. **Actions on $\mathcal{C}_\pm$.** For any real number a, let $\mathcal{A}_{\le a}$ be the space of generalized functions α on $F^\times$ whose support is contained in $\{t \in F^\times \mid v(t) \le a\}$. Let $\mathcal{A}_-$ denote the union of all $\mathcal{A}_{\le a}$ for all a. Then $\mathcal{A}_-$ becomes an algebra under convolution using the measure $d^\times t$.

3.4.1. We have an action of $\mathcal{A}_-$ on $\mathcal{C}_-$ defined by

$$(\alpha * \varphi)(\xi) = \int_{F^\times} \alpha(t)\varphi(t^{-1}\xi)d^\times t$$

for $\alpha \in \mathcal{A}_-$, $\varphi \in \mathcal{C}_-$, and $\xi \in F^n \setminus \{0\}$. One similarly defines $\mathcal{A}_{\ge a}$, $\mathcal{A}_+$, and an action of $\mathcal{A}_+$ on $\mathcal{C}_+$. There is an isomorphism $\sigma : \mathcal{A}_{\le a} \to \mathcal{A}_{\ge -a}$ defined by

$$\sigma(\alpha)(t) = \alpha(t^{-1})|t|^{-n}.$$

3.4.2. We would like to define a multiplicative convolution action of $\mathcal{A}_+$ on $\mathcal{S}'_b(F)$ by

$$(\widetilde{\alpha} * \beta)(s) = \int_{F^\times} \widetilde{\alpha}(t)\beta(t^{-1}s)d^\times t$$

for $\widetilde{\alpha} \in \mathcal{A}_+$ and $\beta \in \mathcal{S}'_b(F)$, but we must explain the meaning of this formula as a distribution on F. Let $\mathcal{S}(F)$ denote the space of locally constant, compactly supported functions on F.

Lemma 3.4.3. *Let $f \in \mathcal{S}(F)$ and $t \in F^\times$. Then $\int_F \beta(t^{-1}s)f(s)ds = 0$ if $v(t)$ is sufficiently large.*

Proof. Since $f \in \mathcal{S}(F)$, there exists an open subgroup $U \subset \mathcal{O}^\times$ that stabilizes f under homotheties. Thus we can replace β by the multiplicative average β_U, which is compactly supported. Then $\int_F \beta(t^{-1}s)f(s)ds = |t| \int_{\operatorname{supp}\beta_U} \beta_U(s)f(ts)ds$. If $v(t)$ is large enough such that f is constant on $t(\operatorname{supp}\beta_U)$, the integral vanishes since $\langle \beta_U, 1\rangle = 0$. $\square$

Define the distribution $\widetilde{\alpha} * \beta \in \mathcal{S}'_b(F)$ by putting the value at $f \in \mathcal{S}(F)$ to be

$$\langle \widetilde{\alpha} * \beta, f\rangle = \int_{F^\times} \widetilde{\alpha}(t) \left(\int_F \beta(t^{-1}s)f(s)ds \right) d^\times t,$$

which is well-defined by Lemma 3.4.3 and the fact that $\widetilde{\alpha} \in \mathcal{A}_+$.

Remark 3.4.4. Observe that $\mathcal{A}_{\le a} * \mathcal{C}_{\le R} \subset \mathcal{C}_{\le R+a}$ and $\mathcal{A}_{\ge a} * \mathcal{C}_{\ge R} \subset \mathcal{C}_{\ge R+a}$ for any numbers a and R. Moreover if $\widetilde{\alpha} \in \mathcal{A}_{\ge a}$ and $\beta \in \mathcal{S}'_b(F)$ has support contained in $\mathfrak{p}^i$, then the support of $\widetilde{\alpha} * \beta$ is contained in $\mathfrak{p}^{a+i}$.

Remark 3.4.5. The convolution action of $\mathcal{A}_+$ on $\mathcal{S}'_b(F)$ is indeed an action, i.e., $\widetilde{\alpha}_1 * (\widetilde{\alpha}_2 * \beta) = (\widetilde{\alpha}_1 * \widetilde{\alpha}_2) * \beta$ for $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathcal{A}_+$ and $\beta \in \mathcal{S}'_b(F)$. One sees this by restricting β to $F^\times$ and identifying $\mathcal{A}_+$ with the space of distributions on $F^\times$ with bounded support using the measure $d^\times t$.

Lemma 3.4.6. *Let $\alpha \in \mathcal{A}_-$, $\beta \in \mathcal{S}'_b(F)$, and $\varphi \in \mathcal{C}_-$. Then*

$$A_\beta(\alpha * \varphi) = \sigma(\alpha) * A_\beta\varphi = A_{\sigma(\alpha)*\beta}(\varphi)$$

Proof. By Corollary 3.2.4 and Remark 3.4.4, we reduce to the case where $\alpha \in \mathcal{A}_- \cap \mathcal{A}_+$ and $\varphi \in \mathcal{C}_c$. Consequently, $\alpha * \varphi \in \mathcal{C}_c$. Fix $x \in F^n \setminus \{0\}$. We have

$$A_\beta(\alpha * \varphi)(x) = \int_{F^n} \beta(\xi \cdot x) \int_{F^\times} \alpha(t)\varphi(t^{-1}\xi) d^\times t d\xi = \int_{F^\times} \alpha(t)|t|^n \int_{F^n} \beta(\xi \cdot tx)\varphi(\xi) d\xi d^\times t.$$

by a change of variables. Substituting t with t^{-1} in the last integral shows that $A_\beta(\alpha * \varphi) = \sigma(\alpha) * A_\beta\varphi$. One observes that $\sigma(\alpha) * A_\beta\varphi = A_{\sigma(\alpha)*\beta}(\varphi)$ essentially by definition. □

Remark 3.4.7. One easily checks that if $\alpha \in \mathcal{A}_-$ and $f \in \mathcal{C}_+$, then $M(\sigma(\alpha) * f) = \alpha * Mf$.

3.5. **Relation between Radon and Fourier transforms.** Note that $\mathcal{F}'$ and M are both operators $\mathcal{C}_+ \to \mathcal{C}_-$. Comparing formulas (2.1) and (3.1), we deduce the formula

$$\mathcal{F}' f = \alpha * Mf \tag{3.2}$$

where $f \in \mathcal{C}_+$ and $\alpha(t) := \psi(-t) - 1$ for $t \in F^\times$.

Let β be the distribution defined in Theorem 3.2.6.

Lemma 3.5.1. *We have an equality of distributions*

$$\beta = \sigma(\alpha) * \psi.$$

Proof. Let $f \in \mathcal{S}(F)$. Then $\langle \sigma(\alpha) * \psi, f \rangle = \int_{F^\times} |t|^n (\psi(t) - 1) \left(\int_F f(s)\psi(-ts) ds \right) d^\times t$. This is the value at f of the Fourier transform of $|t|^{n-1}(\psi(t) - 1)$ considered as a distribution on F. It is well-known [GGP, Ch. 2, §2.5-6] that the Fourier transform of $|t|^{n-1}$ is $\frac{1-q^{n-1}}{1-q^{-n}}|s|^{-n}$. Therefore we conclude that $\sigma(\alpha) * \psi = \beta$. □

Observe that $\mathcal{F} = A_\psi$ and A_β are both operators $\mathcal{C}_- \to \mathcal{C}_+$. Let $\varphi \in \mathcal{C}_-$. From Lemmas 3.4.6 and 3.5.1, we deduce the equality

$$A_\beta \varphi = \sigma(\alpha) * \mathcal{F}\varphi. \tag{3.3}$$

Proof of Theorem 3.2.6. We deduce from (3.2) and Proposition 3.3.1 that M has a left inverse sending $\varphi \in \mathcal{C}_-$ to $\mathcal{F}(\alpha * \varphi)$. Lemma 3.4.6 and (3.3) together say that $\mathcal{F}(\alpha * \varphi) = A_\beta\varphi$. Applying M to (3.3) and using Remark 3.4.7, we see that $MA_\beta = \mathcal{F}'\mathcal{F} = \mathrm{id}$. Therefore A_β is both left and right inverse to M. □

3.6. **Comparison with Černov's Radon inversion formula.** Let f be a Schwartz (i.e., compactly supported C^∞) function on F^n. Recall that the Radon transform $\mathcal{R}f(\xi, s)$ is a C^∞ function on $(F^n \setminus \{0\}) \times F$ (in particular it is defined at $s = 0$), and $\mathcal{R}f(\xi, s) = 0$ if $\|s\xi\|^{-1}$ is sufficiently large. The following "non-archimedean Cavalieri's condition" is also well-known:

Lemma 3.6.1. *The integral $\int_F \mathcal{R}f(\xi, s) ds$ does not depend on ξ.*

Proof. The integral of f over F^n along a pencil of parallel hyperplanes does not depend on the direction of the pencil. □

3.6.2. It was previously known ([Ch, Theorem 5], [Koc, formula (8)]) that the following inversion formula holds:

$$f(x) = \frac{1 - q^{n-1}}{(1 - q^{-1})(1 - q^{-n})} \int_{\|\eta\|=1} \langle |s|^{-n}, \mathcal{R}f(\eta, s + \eta \cdot x) \rangle d\eta \tag{3.4}$$

where $x \in F^n \setminus \{0\}$ and η ranges over norm 1 vectors in F^n.

3.6.3. We will deduce formula (3.4) from Theorem 3.2.6. Since f is compactly supported on F^n, we have $f \in \mathcal{C}_+$ and Theorem 3.2.6 implies that

$$f(x) = A_\beta M f(x) = \int_{v(\xi)\geq R} \beta(\xi \cdot x) M f(\xi) d\xi$$

for $x \in F^n \setminus \{0\}$ and R a sufficiently large number. We can write $\xi = t^{-1}\eta$ where $t \in F^\times$ and $\eta \in F^n$ with $\|\eta\| = 1$. This gives the equality

$$f(x) = \int_{v(t)\leq -R} \int_{\|\eta\|=1} \beta(t^{-1}\eta \cdot x) M f(t^{-1}\eta) |t|^{-n} d\eta d^\times t.$$

Homogeneity of $\mathcal{R}f$ implies that $|t|^{-1} M f(t^{-1}\eta) = \mathcal{R}f(\eta, t)$. Therefore we have the formula

$$f(x) = \frac{1-q^{n-1}}{(1-q^{-1})(1-q^{-n})} \int_{v(t)\leq -R} \int_{\|\eta\|=1} (|\eta \cdot x - t|^{-n} - |\eta \cdot x|^{-n}) \mathcal{R}f(\eta, t) d\eta dt. \tag{3.5}$$

Choose $\eta_0 \in F^n$ with $v(\eta_0) = -v(x)$ and $\eta_0 \cdot x = 1$. Then $\eta \cdot x - t = (\eta - t\eta_0) \cdot x$. Note that if $v(t) > v(x)$, then translation by $t\eta_0$ preserves the unit sphere of norm 1 vectors. Moreover smoothness of $\mathcal{R}f$ implies that $\mathcal{R}f(\eta + t\eta_0, t) = \mathcal{R}f(\eta, t)$ if $v(t)$ is sufficiently large. Therefore the inner integral of (3.5) is zero if $v(t)$ is sufficiently large. Thus we may integrate over all $t \in F$ and switch the order of integration.

Lemma 3.6.4. *The integral $\int_{\|\eta\|=1} |\eta \cdot x|^{-n} d\eta$ equals zero.*

Proof. Using the G-action, we may assume that $x = (1, 0, \ldots, 0)$. Then $\eta \cdot x = \eta_1$, the first coordinate of η. One sees that $\int_{\|\eta\|=1} |\eta_1|^{-n} d\eta = (1 - q^{-1}) + \int_{\mathfrak{p}} |\eta_1|^{-n} d\eta_1 (1 - q^{1-n})$. A simple calculation shows that the latter expression vanishes. □

Lemmas 3.6.1 and 3.6.4 imply that $\int_{\|\eta\|=1} |\eta \cdot x|^{-n} \int_F \mathcal{R}f(\eta, t) dt d\eta = 0$, so the $|\eta \cdot x|^{-n}$ term in (3.5) vanishes. After a change of variables $s = t - \eta \cdot x$, the formula (3.5) becomes equal to Černov's formula (3.4).

4. F real

In this section we prove the invertibility of M when $F = \mathbb{R}$. Recall that in this case $K = O(n)$. The inversion formula is given in Theorem 4.3.3, and a reformulation using the Mellin transform is given in Theorem 4.4.1. The K-finiteness of $\mathcal{C}$ plays a crucial role in the proofs, so we begin by recalling the classification of the K-isotypic components of $\mathcal{C}$.

The non-K-finite situation is considered in §4.7.

4.1. Spherical harmonics.

Let S^{n-1} denote the unit sphere centered at the origin in $\mathbb{R}^n$, which has a natural action by $O(n)$. Let $\mathcal{C}(S^{n-1})$ be the space of smooth K-finite functions on S^{n-1}. For a nonnegative integer k, let H^k denote the space of harmonic polynomials on $\mathbb{R}^n$ of degree k.

Theorem 4.1.1 ([H2, Theorem I.3.1], [JW, Theorem 3.1], [Kos])**.** *Let $H^k|S^{n-1}$ denote the space of harmonic polynomials restricted to S^{n-1}. Then*

(i) the restriction map $H^k \to H^k|S^{n-1}$ is an isomorphism,

(ii) $\mathcal{C}(S^{n-1}) = \bigoplus_{k\geq 0} H^k|S^{n-1}$ as $O(n)$-representations,

(iii) the $O(n)$-representations H^k are irreducible and not isomorphic to each other.

4.2. **Decomposing $\mathcal{C}$ into K-isotypes.** We have a decomposition $\mathbb{R}^n \setminus \{0\} = \mathbb{R}_{>0} \times S^{n-1}$, with $O(n)$ acting on the S^{n-1} component. Let $\mathcal{C}(\mathbb{R}_{>0})$ denote the space of smooth functions on $\mathbb{R}_{>0}$ and define the subspaces $\mathcal{C}_\pm(\mathbb{R}_{>0}), \mathcal{C}_c(\mathbb{R}_{>0})$ as in §2.2.

Theorem 4.1.1 implies that there is a decomposition

$$\mathcal{C} = \bigoplus_{k \geq 0} \mathcal{C}(\mathbb{R}_{>0}) \otimes H^k.$$

For $u \in \mathcal{C}(\mathbb{R}_{>0})$ and $Y \in H^k$, we define $u \otimes Y \in \mathcal{C}$ by $(u \otimes Y)(x) := u(|x|) \cdot Y(\frac{x}{|x|})$.

4.3. **Radon inversion formula.**

4.3.1. We have an isomorphism $\mathrm{Inv} : \mathcal{C}_- \to \mathcal{C}_+$ defined by

$$(\mathrm{Inv}\,\varphi)(x) = \|x\|^{-n} \varphi\left(\frac{x}{\|x\|^2}\right).$$

Set $\widetilde{M} := \mathrm{Inv}^{-1} \circ M$. Consider $\mathbb{R}_{>0}$ as a subgroup of diagonal matrices in G. Then it follows from (2.2) that $\widetilde{M}$ is a $K \times \mathbb{R}_{>0}$ equivariant operator from $\mathcal{C}_+$ to $\mathcal{C}_+$.

4.3.2. Let A denote the space of distributions on $\mathbb{R}_{>0}$ supported on $(0,1]$. Then A is an algebra under the convolution product $*$ induced by the multiplication operation on $\mathbb{R}_{>0}$. The action of $\mathbb{R}_{>0}$ on $\mathcal{C}_+$ induces an action of A.

Theorem 4.3.3. *The operator $M : \mathcal{C}_+ \to \mathcal{C}_-$ is an isomorphism. For $\varphi \in \mathcal{C}_-(\mathbb{R}_{>0}) \otimes H^k$, the inverse $M^{-1} : \mathcal{C}_- \to \mathcal{C}_+$ is given by the formula*

$$M^{-1}\varphi = \beta_k * \mathrm{Inv}(\varphi)$$

where β_k is the distribution on $\mathbb{R}_{>0}$ defined by

$$\beta_k(t) = \frac{1}{2^{n+k-2}\pi^{\frac{n-1}{2}}\Gamma(\frac{n+2k-1}{2})} t^{k-1} \left(-\frac{d}{dt}\right)^{n+k-1} \left(t^{-k+1}(1-t^2)_+^{\frac{n+2k-3}{2}}\right) dt. \tag{4.1}$$

The derivative $\frac{d}{dt}$ is applied in the sense of generalized functions. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -1$, the generalized function $(1-t)_+^\lambda$ is defined by $\langle (1-t)_+^\lambda, f_0(t)dt \rangle = \int_0^1 (1-t)^\lambda f_0(t)dt$ for $f_0 \in \mathcal{C}_c(\mathbb{R}_{>0})$. This generalized function can be analytically continued to all $\lambda \in \mathbb{C}$ not equal to a negative integer [GS, §I.3.2]. We define $(1-t^2)_+^\lambda = (1+t)^\lambda \cdot (1-t)_+^\lambda$.

Corollary 4.3.4. *For any number R one has $M^{-1}(\mathcal{C}_{\leq -R}) \subset \mathcal{C}_{\geq R}$.*

Proof. Observe that β_k is supported on $(0,1]$ for all k. □

4.4. **A formula for $\widetilde{M}$ in terms of convolution.** For $t \in (-1,1)$, define $A_t : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ such that $(A_t f)(x)$ is the average value of f on the $(n-2)$-sphere $\{\omega \in S^{n-1} \mid \omega \cdot x = t\}$. Then A_t is $O(n)$-equivariant, so by Schur's lemma it acts on $H^k|S^{n-1}$ by a scalar $a_k(t)$. Since H^k is stable under complex conjugation, $a_k(t)$ is real valued. One observes that a_k is a smooth function on $(-1,1)$, $|a_k(t)| \leq 1$ for all $t \in (-1,1)$, and $\lim_{t \to 1} a_k(t) = 1$.

Suppose that $f \in \mathcal{C}(\mathbb{R}_{>0}) \otimes H^k$ and there exists $C > 0$ and $\sigma > n-1$ such that $|f(x)| \leq C\|x\|^{-\sigma}$ for all x with $\|x\| \geq 1$. Since the intersection of S^{n-1} with the hyperplane $\{\omega \mid \omega \cdot x = t\}$ has radius $(1-t^2)^{1/2}$ for a unit vector x, we deduce that

$$\widetilde{M} f = \alpha_k * f \tag{4.2}$$

where α_k is the measure $\mathrm{mes}(S^{n-2}) \cdot t^{-n} \cdot a_k(t)(1-t^2)^{\frac{n-3}{2}} dt$ on the interval $(0,1)$ extended by zero to the whole $\mathbb{R}_{>0}$. The convolution $\alpha_k * f$ is well-defined because of the bound on $|f(x)|$, and $\mathrm{mes}(S^{n-2})$ denotes the surface area of the $(n-2)$-sphere. In fact, [H1, Proposition 2.11] says that $a_k(t)$ is the scalar multiple of the Gegenbauer polynomial $C_k^{(\frac{n-2}{2})}(t)$ normalized by $a_k(1) = 1$.

The Mellin transform $\mathfrak{M}\alpha_k$ is defined for $s \in \mathbb{C}$ by integrating t^s against α_k if $\mathrm{Re}(s) > n-1$.

Theorem 4.4.1. *The distribution α_k is invertible in A. The inverse β_k is defined by (4.1). The Mellin transforms are given by*

$$\mathfrak{M}\beta_k(s) = \frac{1}{\mathfrak{M}\alpha_k(s)} = 2^{1-n-k}\pi^{\frac{1-n}{2}} \frac{\Gamma(s+k)}{\Gamma(s-n+1)} \cdot \frac{\Gamma(\frac{s-n-k}{2}+1)}{\Gamma(\frac{s+k+1}{2})}. \tag{4.3}$$

Theorem 4.4.1 implies Theorem 4.3.3.

4.5. **Relation to Fourier transform.** Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Fourier transform is defined for an integrable function f on $\mathbb{R}^n$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx.$$

This definition can be extended [SW, §I.3] to the space of tempered distributions. After this extension, $\mathcal{F}$ becomes an isomorphism $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Let $F : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ denote the 1-dimensional Fourier transform. For $f \in \mathcal{S}(\mathbb{R}^n)$, one gets the Fourier transform from the Radon transform by

$$\mathcal{F}f(r\omega) = F(\mathcal{R}f(\omega, t))(r) \tag{4.4}$$

where F is the Fourier transform with respect to the t variable, $r \in \mathbb{R}$, and $\omega \in S^{n-1}$ is a unit vector.

Lemma 4.5.1. *Let f be a locally integrable function on $\mathbb{R}^n$ for which there exist $C > 0$ and $\sigma > n-1$ such that $|f(x)| \le C\|x\|^{-\sigma}$ for all x with $\|x\| \ge 1$. Then:*
(i) $\mathcal{R}f$ is a locally integrable function on $S^{n-1} \times \mathbb{R}$.
(ii) $\mathcal{R}f(\omega, t)$ is bounded for $|t| \ge 1$.
(iii) The right hand side of (4.4) is well-defined as a generalized function on $\mathbb{R} \times S^{n-1}$.
(iv) Equation (4.4) holds as an equality between generalized functions on $\mathbb{R}_{>0} \times S^{n-1}$.

Proof. Since $\mathcal{R}f$ is defined by integrating f on a hyperplane of dimension $n-1$, the bound on $|f(x)|$ implies that $\mathcal{R}f$ is well-defined on $S^{n-1} \times \mathbb{R}$. One also uses this bound and local integrability of f to deduce that $\mathcal{R}f$ is locally integrable. If $\omega \in S^{n-1}$ and $t \in \mathbb{R}$ with $|t| \ge 1$, then integrating in the radial direction on the hyperplane $\omega \cdot x = t$, we see that $|\mathcal{R}f(\omega, t)|$ is bounded by a constant times $\int_0^\infty (r^2 + t^2)^{-\sigma/2} r^{n-2} dr$, which is equal to a constant times $|t|^{n-1-\sigma}$. This proves (ii). Property (iii) follows immediately from properties (i)-(ii).

Let φ be a compactly supported smooth function on $\mathbb{R}^n \setminus \{0\} = \mathbb{R}_{>0} \times S^{n-1}$. Consider f as a tempered distribution on $\mathbb{R}^n$. By the definition of $\mathcal{F}f$,

$$\int_{\mathbb{R}_{>0} \times S^{n-1}} \mathcal{F}f(r\omega)\varphi(r\omega) r^{n-1} dr d\omega = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}_{>0} \times S^{n-1}} \varphi(r\omega) e^{-2\pi i r(\omega \cdot x)} r^{n-1} dr d\omega dx. \tag{4.5}$$

Since $t \mapsto \int_{\mathbb{R}_{>0}} \varphi(r\omega) e^{-2\pi i r t} r^{n-1} dr$ is a Schwartz function on $\mathbb{R}$, we deduce from the decomposition $dx = d\mu_\omega dt$ and property (ii) applied to $|f|$ that the integral

$$\int_{S^{n-1}} \int_{\mathbb{R}^n} \left| f(x) \int_{\mathbb{R}_{>0}} \varphi(r\omega) e^{-2\pi i r(\omega \cdot x)} r^{n-1} dr \right| dx d\omega$$

converges. Then the Fubini-Tonelli theorem implies that (4.5) is equal to

$$\int_{S^{n-1}}\int_{\mathbb{R}}\int_{\mathbb{R}_{>0}} \mathcal{R}f(\omega,t)e^{-2\pi i r t}\varphi(r\omega)r^{n-1}drdtd\omega,$$

which proves (iv). □

4.6. **Proof of Theorem 4.4.1.** Let $Y \in H^k$ and define $f(x) = \|x\|^{-s}\cdot Y(\frac{x}{\|x\|})$ for $s \in \mathbb{C}$. If $n-1 < \mathrm{Re}(s) < n$, then f is locally integrable on $\mathbb{R}^n$ and satisfies the hypothesis of Lemma 4.5.1. Moreover by (4.2) and homogeneity of f we see that

$$\mathcal{R}f(\omega,t) = \mathrm{sgn}(t)^k|t|^{n-1-s}Mf(\omega) = \mathrm{sgn}(t)^k|t|^{n-1-s}\mathfrak{M}\alpha_k(s)Y(\omega)$$

as a locally integrable function on $S^{n-1}\times\mathbb{R}$. Then Lemma 4.5.1 implies that

$$\mathcal{F}f(r\omega) = F(\mathrm{sgn}(t)^k|t|^{n-1-s})(r)\mathfrak{M}\alpha_k(s)Y(\omega).$$

It is well-known [GS, §II.2.3] that

$$F(\mathrm{sgn}(t)^k|t|^{n-1-s})(r) = i^k(2\pi)^{s-n+1}\frac{\sin(\pi\frac{(s-n-k+1)}{2})}{\pi}\Gamma(n-s)\,\mathrm{sgn}(r)^k|r|^{s-n}.$$

On the other hand, one can compute the Fourier transform of f directly:

Theorem 4.6.1 ([SW, Theorem IV.4.1])**.** *If $0 < \mathrm{Re}(s) < n$, then $\mathcal{F}f(x) = \gamma\|x\|^{s-n}Y(\frac{x}{\|x\|})$, where $\gamma = i^{-k}\pi^{s-\frac{n}{2}}\Gamma(\frac{n+k-s}{2})/\Gamma(\frac{s+k}{2})$.*

Comparing constant multiples in the two formulas for $\mathcal{F}f$ above and applying Euler's reflection formula, we have

$$\mathfrak{M}\alpha_k(s) = 2^{n-1-s}\pi^{n/2-1}\frac{\Gamma(\frac{n+k-s}{2})\Gamma(\frac{s-n-k+1}{2})\Gamma(\frac{n+k+1-s}{2})}{\Gamma(\frac{s+k}{2})\Gamma(n-s)}$$

for $n-1 < \mathrm{Re}(s) < n$. By analytic continuation, we deduce the equality for all $s \in \mathbb{C}$ away from poles. The duplication formula for the Γ-function implies that

$$\mathfrak{M}\alpha_k(s) = 2^{n+k-1}\pi^{\frac{n-1}{2}}\frac{\Gamma(s-n+1)}{\Gamma(s+k)}\cdot\frac{\Gamma(\frac{s+k+1}{2})}{\Gamma(\frac{s-n-k}{2}+1)}, \tag{4.6}$$

as stated in Theorem 4.4.1. To finish the proof of Theorem 4.4.1, it remains to show that $(\mathfrak{M}\beta_k)^{-1}$ equals the right hand side of (4.6). By considering the Beta function we see that $\Gamma(\frac{s-n-k}{2}+1)/\Gamma(\frac{s+k+1}{2})$ is the Mellin transform of $\nu(t)dt$, where

$$\nu(t) = \frac{2}{\Gamma(\frac{n+2k-1}{2})}t^{1-n-k}(1-t^2)_+^{\frac{n+2k-3}{2}}. \tag{4.7}$$

The generalized function $(1-t^2)_+^{\frac{n+2k-3}{2}}$ is defined in the paragraph after Theorem 4.3.3. Multiplying the right hand side of (4.7) by $\Gamma(s+k)/\Gamma(s-n+1) = (s-n+1)\cdots(s+k-1)$ amounts to replacing ν by $L_k(\nu)$, where $L_k := (-\frac{d}{dt}\cdot t-n+1)\cdots(-\frac{d}{dt}\cdot t+k-1)$. Observe that $L_k = t^{k-1}(-\frac{d}{dt})^{n+k-1}t^n$. Therefore $\mathfrak{M}\beta_k = (\mathfrak{M}\alpha_k)^{-1}$, where β_k is defined by (4.1). This proves Theorem 4.4.1.

In the case $n=2$, the formula (4.6) is well-known (cf. [Wal, Lemma 7.17], [Bu, Proposition 2.6.3]).

4.7. **The non-K-finite situation.** In this subsection we consider the situation where we remove K-finiteness from the definitions of $\mathcal{C}_+$ and $\mathcal{C}_-$. Let $\mathscr{C}_+$ be the space of smooth functions on $\mathbb{R}^n\setminus\{0\}$ with bounded support, and let $\mathscr{C}_-$ be the space of smooth functions on $(\mathbb{R}^n)^*\setminus\{0\}$ supported away from a neighborhood of 0.

4.7.1. We have the operator $\mathscr{M} : \mathscr{C}_+ \to \mathscr{C}_-$ defined by

$$\mathscr{M} f(\xi) = \int_{\langle \xi, x\rangle = 1} f(x) d\mu_\xi$$

(cf. formula (2.1)). One can deduce that $\mathscr{M}$ is injective from the injectivity of $M : \mathcal{C}_+ \to \mathcal{C}_-$. However we will show below that $\mathscr{M}$ is not surjective, and hence not an isomorphism.

4.7.2. Let $f \in \mathscr{C}_+$. Define $C_f = \operatorname{supp}(f) \cup \{0\}$, which is a compact subset of $\mathbb{R}^n$. Let $\widehat{C}_f$ denote its convex hull.

Let $C \subset \mathbb{R}^n$ be a convex set containing 0. Define $C^* \subset (\mathbb{R}^n)^*$ to be the set of ξ such that the hyperplane $\langle \xi, x\rangle = 1$ is disjoint from C. By convexity,

$$C^* = \{\xi \mid \langle \xi, x\rangle < 1 \text{ for all } x \in C\}.$$

Observe that C^* is a convex set[3] containing 0. If $\check{C} \subset (\mathbb{R}^n)^*$ is a convex set containing 0, one similarly defines the dual $\check{C}^* \subset \mathbb{R}^n$. Taking duals gives mutually inverse maps between the collection of compact convex subsets of $\mathbb{R}^n$ containing 0 and the collection of open convex subsets of $(\mathbb{R}^n)^*$ containing 0.

Proposition 4.7.3. *The connected component of* $(\mathbb{R}^n)^* \setminus \operatorname{supp}(\mathscr{M} f)$ *containing* 0 *is equal to* $(\widehat{C}_f)^*$. *In particular, it is convex.*

Corollary 4.7.4. *The operator* $\mathscr{M} : \mathscr{C}_+ \to \mathscr{C}_-$ *is not surjective.*

Lemma 4.7.5. *Let* $\xi_0 \in (\mathbb{R}^n)^* \setminus \{0\}$. *If* $\mathscr{M} f$ *vanishes on a neighborhood of the segment* $[0, \xi_0] := \{t\xi_0 \mid 0 \le t \le 1\}$, *then* f *vanishes on the half-space* $\langle \xi_0, x\rangle \ge 1$.

Proof. By replacing f by a compactly supported function that is equal to f outside of a small neighborhood of 0, we may assume that f is compactly supported. There exists an open convex neighborhood $\check{C}$ of $[0, \xi_0]$ such that $\mathscr{M} f$ vanishes on $\check{C}$. Then $C = \check{C}^*$ is a compact convex subset of $\mathbb{R}^n$ and $C^* = \check{C}$, so the integral of f along any hyperplane disjoint from C vanishes. Therefore [H3, Corollary 2.8] implies that $\operatorname{supp} f \subset C$. Since $\xi_0 \in \check{C}$, one sees that C is contained in the half-space $\langle \xi_0, x\rangle < 1$. □

We have the support function $H : (\mathbb{R}^n)^* \to \mathbb{R}$ associated to C_f, which is defined by

$$H(\xi) = \sup\{\langle \xi, x\rangle \mid x \in C_f\}.$$

For $\xi \ne 0$, the set $\{x \mid \langle \xi, x\rangle = H(\xi)\}$ is a supporting hyperplane of $\widehat{C}_f$. The function H uniquely determines the compact convex set $\widehat{C}_f$, and $(\widehat{C}_f)^* = H^{-1}(\mathbb{R}_{<1})$.

Proof of Proposition 4.7.3. It is clear that $H^{-1}(\mathbb{R}_{<1})$ is an open subset of $(\mathbb{R}^n)^* \setminus \operatorname{supp}(\mathscr{M} f)$. Note that since $\operatorname{supp}(\mathscr{M} f)$ is closed, Lemma 4.7.5 implies that if $H(\xi) = 1$ then $\xi \in \operatorname{supp}(\mathscr{M} f)$. Thus $(\widehat{C}_f)^* = H^{-1}(\mathbb{R}_{<1})$ is also closed in $(\mathbb{R}^n)^* \setminus \operatorname{supp}(\mathscr{M} f)$. □

5. F complex

In this section we prove the invertibility of M when $F = \mathbb{C}$. Recall that in this case $K = U(n)$. The inversion formula is given in Theorem 5.3.3, and a reformulation using the Mellin transform is given in Theorem 5.4.1. The K-finiteness of $\mathcal{C}$ plays a crucial role in the proofs, so we begin by recalling the classification of the K-isotypic components of $\mathcal{C}$.

[3] C^* is called [Ca] the dual (polar) set of C. Note that if C is compact, then C^* is open. If 0 is an interior point of C, then C^* is bounded.

5.1. **Spherical harmonics.** Let S^{2n-1} denote unit sphere of norm 1 vectors in $\mathbb{C}^n = \mathbb{R}^{2n}$, which has a natural action by $U(n)$. Let $\mathcal{C}(S^{2n-1})$ be the space of smooth K-finite functions on S^{2n-1}. For nonnegative integers p, q, let $H^{p,q}$ denote the homogeneous polynomials of degree $p+q$ on $\mathbb{R}^{2n}$ that are harmonic and satisfy

$$Y(\lambda z_1, \dots, \lambda z_n) = \lambda^p \overline{\lambda}^q Y(z_1, \dots, z_n)$$

for $\lambda \in \mathbb{C}, (z_1, \dots, z_n) \in \mathbb{C}^n = \mathbb{R}^{2n}$.

Theorem 5.1.1 ([JW, Theorem 3.1], [Kos])**.** *Let $H^{p,q}|S^{2n-1}$ denote the space of harmonic polynomials restricted to S^{2n-1}. Then*

(i) $\mathcal{C}(S^{2n-1}) = \bigoplus_{p,q\geq 0} H^{p,q}|S^{2n-1}$ as $U(n)$-representations,

(ii) the $U(n)$-representations $H^{p,q}$ are irreducible and not isomorphic to each other.

5.2. **Decomposing $\mathcal{C}$ into K-isotypes.** We have a decomposition $\mathbb{C}^n \setminus \{0\} = \mathbb{R}_{>0} \times S^{2n-1}$, with $O(2n)$ (and hence $U(n)$) acting on the S^{2n-1} component. Let $\mathcal{C}(\mathbb{R}_{>0})$ denote the space of smooth functions on $\mathbb{R}_{>0}$ and define the subspaces $\mathcal{C}_\pm(\mathbb{R}_{>0}), \mathcal{C}_c(\mathbb{R}_{>0})$ as in §2.2.

Theorem 5.1.1 implies that there is a decomposition

$$\mathcal{C} = \bigoplus_{p,q\geq 0} \mathcal{C}(\mathbb{R}_{>0}) \otimes H^{p,q}.$$

For $u \in \mathcal{C}(\mathbb{R}_{>0})$ and $Y \in H^{p,q}$, we define $u \otimes Y \in \mathcal{C}$ by $(u \otimes Y)(x) := u(\|x\|) \cdot Y(\frac{x}{\|x\|})$.

5.3. **Radon inversion formula.**

5.3.1. We have an isomorphism $\mathrm{Inv} : \mathcal{C}_- \to \mathcal{C}_+$ defined by

$$(\mathrm{Inv}\,\varphi)(x) = \|x\|^{-2n} \varphi\left(\frac{\overline{x}}{\|x\|^2}\right)$$

where $\overline{x}$ is coordinate-wise conjugation. Set $\widetilde{M} := \mathrm{Inv}^{-1} \circ M$. Consider $\mathbb{R}_{>0}$ as a subgroup of diagonal matrices in G. Then it follows from (2.2) that $\widetilde{M}$ is a $K \times \mathbb{R}_{>0}$ equivariant operator from $\mathcal{C}_+$ to $\mathcal{C}_+$.

5.3.2. Let A be the space of distributions on $\mathbb{R}_{>0}$ supported on $(0, 1]$ (see §4.3.2). The action of $\mathbb{R}_{>0}$ on $\mathcal{C}_+$ induces an action of A.

Theorem 5.3.3. *The operator $M : \mathcal{C}_+ \to \mathcal{C}_-$ is an isomorphism. For $\varphi \in \mathcal{C}_-(\mathbb{R}_{>0}) \otimes H^{p,q}$, the inverse $M^{-1} : \mathcal{C}_- \to \mathcal{C}_+$ is given by the formula*

$$M^{-1}\varphi = \beta_{p,q} * \mathrm{Inv}(\varphi)$$

where $\beta_{p,q}$ is the distribution on $\mathbb{R}_{>0}$ defined by

$$\beta_{p,q}(t) = \frac{1}{2^{n+m-2}\pi^{n-1}\Gamma(m)} \prod_{j=1}^{n+m-1} \left(-\frac{d}{dt} \cdot t + p + q - 2j\right) \left(t^{-p-q-2n+1}(1-t^2)_+^{m-1}\right) dt \tag{5.1}$$

where $m = \min(p, q)$.

The derivative $\frac{d}{dt}$ is applied in the sense of generalized functions. The generalized function $(1-t^2)_+^\lambda$ is defined by analytic continuation for $\lambda \in \mathbb{C}$ (see the paragraph following the statement of Theorem 4.3.3). In particular, the regularization of $\frac{2}{\Gamma(m)}(1-t^2)_+^{m-1}dt$ at $m = 0$ is equal to $\delta(1-t)$.

Corollary 5.3.4. *For any number R one has $M^{-1}(\mathcal{C}_{\leq -R}) \subset \mathcal{C}_{\geq R}$.*

Proof. Observe that $\beta_{p,q}$ is supported on $(0, 1]$ for all p, q. $\square$

5.4. **A formula for $\widetilde{M}$ in terms of convolution.** We consider the dot product on $S^{2n-1} \subset \mathbb{C}^n$ induced by the dot product on $\mathbb{C}^n$. For $t \in (-1,1)$, define $A_t : \mathcal{C}(S^{2n-1}) \to \mathcal{C}(S^{2n-1})$ such that $(A_t f)(x)$ is the average value of f on the $(2n-3)$-sphere $\{\omega \in S^{2n-1} \mid \omega \cdot \overline{x} = t\}$. Then A_t is $U(n)$-equivariant, so by Schur's lemma it acts on $H^{p,q}|S^{2n-1}$ by a scalar $a_{p,q}(t)$. One observes that $a_{p,q}$ is a smooth function on $(-1,1)$, $|a_{p,q}(t)| \leq 1$ for all $t \in (-1,1)$, and $\lim_{t \to 1} a_{p,q}(t) = 1$.

Suppose that $f \in \mathcal{C}(\mathbb{R}_{>0}) \otimes H^{p,q}$ and there exist $C > 0$ and $\sigma > n-1$ such that $|f(x)| \leq C\|x\|^{-2\sigma}$ for all x with $\|x\| \geq 1$. One can deduce as in the real case that

$$\widetilde{M} f = \alpha_{p,q} * f \tag{5.2}$$

where $\alpha_{p,q}$ is the measure $\operatorname{mes}(S^{2n-3}) \cdot t^{1-2n} \cdot a_{p,q}(t)(1-t^2)^{n-2} dt$ on the interval $(0,1)$ extended by zero to the whole $\mathbb{R}_{>0}$. The convolution $\alpha_{p,q} * f$ is well-defined due to the bound on $|f(x)|$, and $\operatorname{mes}(S^{2n-3})$ denotes the surface area of the $(2n-3)$-sphere. By considering zonal spherical functions, one can check [Wat, Lemma 1.2] that $a_{p,q}(t)$ is the scalar multiple of the Jacobi polynomial $P^{(n-2,|p-q|)}_{\min(p,q)}(2t^2-1)$ normalized by $a_{p,q}(1) = 1$.

The Mellin transform $\mathfrak{M}\alpha_{p,q}$ is defined for $s \in \mathbb{C}$ by integrating t^s against $\alpha_{p,q}$ if $\operatorname{Re}(s) > 2n-2$.

Theorem 5.4.1. *The distribution $\alpha_{p,q}$ is invertible in A. The inverse $\beta_{p,q}$ is defined by* (5.1)*. The Mellin transforms are given by*

$$\mathfrak{M}\beta_{p,q}(s) = \frac{1}{\mathfrak{M}\alpha_{p,q}(s)} = \pi^{1-n} \frac{\Gamma(\frac{s+p+q}{2})}{\Gamma(\frac{s+|p-q|}{2} - n + 1)} \cdot \frac{\Gamma(\frac{s-p-q}{2} - n + 1)}{\Gamma(\frac{s-|p-q|}{2} - n + 1)}. \tag{5.3}$$

Theorem 5.4.1 implies Theorem 5.3.3.

5.5. **Relation to the Fourier transform.** Let $\mathcal{S}(\mathbb{C}^n)$ denote the space of Schwartz functions on $\mathbb{C}^n$ and $\mathcal{S}'(\mathbb{C}^n)$ the dual space of tempered distributions on $\mathbb{C}^n$. The Fourier transform is defined for an integrable function f on $\mathbb{C}^n$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{C}^n} f(x) e^{-2\pi i \operatorname{Re}(\xi \cdot \overline{x})} dx.$$

This definition coincides with the one from §4.5 by identifying $\mathbb{C}^n = \mathbb{R}^{2n}$. The Fourier transform can be extended to an isomorphism of tempered distributions $\mathcal{F} : \mathcal{S}'(\mathbb{C}^n) \to \mathcal{S}'(\mathbb{C}^n)$.

Let $F : \mathcal{S}'(\mathbb{C}) \to \mathcal{S}'(\mathbb{C})$ denote the Fourier transform over $\mathbb{C}$. For $f \in \mathcal{S}(\mathbb{C}^n)$, one gets $\mathcal{F}f$ from the Radon transform by

$$\mathcal{F}f(r\omega) = F(\mathcal{R}f(\overline{\omega}, t))(r) \tag{5.4}$$

where F is the Fourier transform with respect to the t variable, $r \in \mathbb{C}$, and $\omega \in S^{2n-1}$ is a unit vector. We have the following complex analog of Lemma 4.5.1, which is proved in exactly the same way.

Lemma 5.5.1. *Let f be a locally integrable function on $\mathbb{C}^n$ for which there exist $C > 0$ and $\sigma > n-1$ such that $|f(x)| \leq C\|x\|^{-2\sigma}$ for all x with $\|x\| \geq 1$. Then:*

(i) $\mathcal{R}f$ is a locally integrable function on $S^{2n-1} \times \mathbb{C}$.

(ii) $\mathcal{R}f(\omega, t)$ is bounded for $|t| \geq 1$.

(iii) The right hand side of (5.4) *is well-defined as a generalized function on $\mathbb{C} \times S^{2n-1}$.*

(iv) Equation (5.4) *holds as an equality between generalized functions on $\mathbb{R}_{>0} \times S^{2n-1}$.*

5.6. **Proof of Theorem 5.4.1.** Let $Y \in H^{p,q}$ and define $f(x) = \|x\|^{-s} \cdot Y(\frac{x}{\|x\|})$ for $s \in \mathbb{C}$. If $2n-2 < \mathrm{Re}(s) < 2n$, then f is locally integrable on $\mathbb{C}^n$ and satisfies the hypothesis of Lemma 5.5.1. Moreover by (5.2) and homogeneity of f we see that

$$\mathcal{R}f(\overline{\omega}, t) = t^p \overline{t}^q |t|^{2n-2-p-q-s} \widetilde{M} f(\omega) = t^p \overline{t}^q |t|^{2n-2-p-q-s} \mathfrak{M}\alpha_{p,q}(s) Y(\omega)$$

as a locally integrable function on $S^{2n-1} \times \mathbb{C}$. Then Lemma 5.5.1 implies that

$$\mathcal{F}f(r\omega) = F(t^p \overline{t}^q |t|^{2n-2-p-q-s})(r) \mathfrak{M}\alpha_{p,q}(s) Y(\omega)$$

as generalized functions on $\mathbb{R}_{>0} \times S^{2n-1}$.

Lemma 5.6.1. *If $2n-2 < \mathrm{Re}(s) < 2n$, then*

$$F(t^p \overline{t}^q |t|^{2n-2-p-q-s})(r) = \pi^{s-2n+1} i^{-|p-q|} \frac{\Gamma(\frac{2n+|p-q|-s}{2})}{\Gamma(\frac{s-2n+2+|p-q|}{2})} r^p \overline{r}^q |r|^{s-2n-p-q}$$

as locally integrable functions on $\mathbb{C}$.

Proof. Apply Theorem 4.6.1 for $n=2$, $k=|p-q|$, and $Y(x_1,x_2) = (x_1+ix_2)^{p-q}$ if $p \geq q$ or $Y(x_1,x_2) = (x_1-ix_2)^{q-p}$ if $p \leq q$. $\square$

Alternatively, we can use Theorem 4.6.1 to find that $\mathcal{F}f(x) = \gamma \|x\|^{s-2n} Y(\frac{x}{\|x\|})$, where $\gamma = i^{-p-q} \pi^{s-n} \Gamma(\frac{2n+p+q-s}{2}) / \Gamma(\frac{s+p+q}{2})$. Comparing the two formulas we have derived for $\mathcal{F}f$ and applying Euler's reflection formula, we conclude that

$$\mathfrak{M}\alpha_{p,q}(s) = \pi^{n-1} \frac{\Gamma(\frac{s+|p-q|}{2} - n + 1)\Gamma(\frac{s-|p-q|}{2} - n + 1)}{\Gamma(\frac{s+p+q}{2})\Gamma(\frac{s-p-q}{2} - n + 1)}, \tag{5.5}$$

as stated in Theorem 5.4.1. The equation holds *a priori* for $2n-2 < \mathrm{Re}(s) < 2n$, and we deduce by analytic continuation that it holds for all $s \in \mathbb{C}$, away from poles.

To finish the proof of Theorem 5.4.1, it remains to show that $(\mathfrak{M}\beta_{p,q})^{-1}$ is equal to the right hand side of (5.5). By considering the Beta function we see that $\Gamma(\frac{s-p-q}{2} - n + 1)/\Gamma(\frac{s-|p-q|}{2} - n + 1)$ is the Mellin transform of $\nu(t)dt$, where

$$\nu(t) = \frac{2}{\Gamma(m)} t^{-p-q-2n+1} (1-t^2)_+^{m-1} \tag{5.6}$$

for $m = \min(p,q)$. Note that if $m=0$, then $\nu(t)dt = \delta(1-t)$. Multiplying the right hand side of (5.6) by $\Gamma(\frac{s+p+q}{2})/\Gamma(\frac{s+|p-q|}{2} - n + 1) = \prod_{j=1}^{n+m-1}(\frac{s+p+q}{2} - j)$ amounts to replacing ν by $L_{p,q}(\nu)$, where $L_{p,q}$ is the differential operator

$$2^{1-n-m} \prod_{j=1}^{n+m-1} \left(-\frac{d}{dt} \cdot t + p + q - 2j \right).$$

Theorem 5.4.1 is proved.

In the case $n=2$, the formula (5.5) is well-known (cf. [Wal, Lemma 7.23], [Du, Proposition III.3.7]).

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