QUOTIENTS OF ALGEBRAIC GROUPS

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1. Introduction

In this note, we study the existence and structure of the homogeneous space G/H for algebraic groups $H \subset G$. Let k be a field. All schemes considered will be k-schemes. By an affine algebraic group, we mean an affine group scheme of finite type over k. Note that we do not assume our schemes are reduced yet. We will only consider affine algebraic groups. From now on, G will denote an algebraic group unless otherwise stated.

1.1. **Two notions of a quotient.** First, we must specify what properties we are looking for in a quotient. The following is taken from [MFK94].

Definition 1.1.1. Let X be a scheme with G-action. A *categorical quotient* is a scheme Y with G-invariant map $X \to Y$ satisfying the universal property

$$\mathrm{Hom}(Y, Z) \simeq \mathrm{Hom}_{G\text{-inv}}(X, Z)$$

is an isomorphism.

Definition 1.1.2. Let X be a scheme with G-action. A *geometric quotient* is a scheme Y with G-invariant map $\pi : X \to Y$ such that:

(1) The natural map $G \times X \to X \times_Y X$ is an isomorphism, and π is surjective on sets.
(2) A subset $U \subset Y$ is open iff $\pi^{-1}(U) \subset X$ is open.
(3) For open $U \subset Y$ and section $f \in \Gamma(\pi^{-1}(U), \mathcal{O}_X)$, we have $f \in \Gamma(U, \mathcal{O}_Y)$ iff

$$\begin{array}{ccc} G \times \pi^{-1}(U) & \xrightarrow{\text{act}} & \pi^{-1}(U) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ \pi^{-1}(U) & \xrightarrow{\quad f \quad} & \mathbf{A}^1 \end{array}$$

commutes.

Lemma 1.1.3 ([MFK94, Proposition 0.1])**.** *A geometric quotient is also a categorical quotient.*

Proof. Assume $\pi : X \to Y$ satisfies the conditions of 1.1.2. Take G-invariant map $\phi : X \to Z$. By 1.1.2(1), π is surjective, so we define $\psi : Y \to Z$ on sets in the obvious way. Then ψ is continuous on topological spaces by 1.1.2(2), and we have $\phi = \psi \circ \pi$. Lastly we need to define $\mathcal{O}_Z \to \psi_* \mathcal{O}_Y$. Since ϕ is G-invariant, $\mathcal{O}_Z \to \phi_* \mathcal{O}_X$ has image in $\psi_*(\mathcal{O}_Y \hookrightarrow \pi_* \mathcal{O}_Z)$ by 1.1.2(3). Thus there is a unique map $\psi : Y \to Z$ of ringed spaces such that $\phi = \psi \circ \pi$. Using this composition and

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π surjective on sets, we deduce that ψ is a map of locally ringed spaces and hence the unique map of schemes desired. $\square$

2. General case

2.1. Quotient as a sheaf. Let G be an algebraic group and H a closed subgroup of G, which does not need to be reduced. We can consider H, G as sheaves on the site $\mathbf{Sch}_{fppf}(k)$. Then we have a presheaf of sets

$$Q : S \mapsto G(S)/H(S).$$

Let G/H denote the sheafification of Q on $\mathbf{Sch}_{fppf}(k)$. Since $G \to Q$ is an epimorphism of presheaves, $G \to G/H$ is an epimorphism of fppf sheaves.

We claim that Q is a separated presheaf. Take fppf covering $S' \to S$. It suffices to show that if $g \in G(S)$ has $g|_{S'} \in H(S') \subset G(S')$, then $g \in H(S)$. Since g satisfies the equalizer condition for G, it also does for $H \subset G$. Thus by the sheaf property of H, we must have $g \in H(S)$. This implies $Q \hookrightarrow G/H$ is a monomorphism of presheaves. The map $G \to G/H$ factors through $G \to Q \hookrightarrow G/H$; thus $G \times_Q G \simeq G \times_{G/H} G$. It is clear that the natural map $G \times H \to G \times_Q G$ is an isomorphism. Thus we have

$$G \times H \simeq G \underset{Q}{\times} G \simeq G \underset{G/H}{\times} G$$

as fppf sheaves.[1]

We wish to show the following:

Theorem 2.1.1. *Let $H \subset G$ be as stated. The fppf sheaf G/H is representable by a quasi-projective scheme with G-equivariant (very) ample line bundle.* A posteriori*, the map of schemes $G \to G/H$ is a categorical quotient with respect to the right H-action on G, fppf, and satisfies 1.1.2(1)(2).*

Our main proof idea follows [DG70, III, §3, Theorem 5.4].

2.2. A theorem of Chevalley. Let I be the ideal corresponding to H, so $\mathcal{O}_H \simeq \mathcal{O}_G/I$. Since everything is of finite type, hence Noetherian, I is finitely generated.

The following lemma is modified from [Hum75, Lemma 8.5].

Lemma 2.2.1. *For any k-algebra A, we have $g \in G(A)$ lies in $H(A)$ iff $\rho_g : k[G] \otimes A \to k[G] \otimes A$ stabilizes $I \otimes A$.*

Here the notation is $G(A) := G(\operatorname{Spec} A) = \operatorname{Hom}(\operatorname{Spec} A, G)$. For $g \in G(A)$, we define ρ_g to be the ring homomorphism corresponding to

$$(1) \qquad G \times \operatorname{Spec} A \xrightarrow{(g',a) \mapsto (g', g(a), a)} G \times G \times \operatorname{Spec} A \xrightarrow{\quad m \quad} G \times \operatorname{Spec} A$$

where the notation may be made rigorous via functor of points.

Proof. Given $g \in G(A) \simeq \operatorname{Hom}(k[G], A)$, we want to see if $k[G] \xrightarrow{g} A$ vanishes on I. Writing down the ring homomorphism corresponding to (1), we have

$$(2) \qquad \rho_g : k[G] \otimes A \xrightarrow{m^* \otimes \mathrm{id}} k[G] \otimes k[G] \otimes A \xrightarrow{\mathrm{id} \otimes g \cdot \mathrm{id}} k[G] \otimes A.$$

[1]This argument is a special case of a more general bijection between certain surjections of fppf-sheaves and equivalence relations in schemes, which ties into the theory of algebraic spaces.

Now we have identity $e \in H(A)$, so $k[G] \xrightarrow{e} A$ dies on I. If $\rho_g(I \otimes A) \subset I \otimes A$, we should have $(e \cdot \mathrm{id}) \circ \rho_g(I \otimes A) = 0$. But if we let $\pi^* : k[G] \hookrightarrow k[G] \otimes A$, we in fact have $(e \cdot \mathrm{id}) \circ \rho_g \circ \pi^* = g$. Thus $g(I) = 0$, as desired.

For the other direction, if $g \in H(A)$, then ρ_g induces a map $A \otimes k[G]/I \to A \otimes k[G]/I$, so $\rho_g(I \otimes A) \subset I \otimes A$. □

We present (again a modified version) of [Hum75, Theorem 11.2]. This theorem may also be found in [DG70, II, §2, Corollaire 3.5].

Theorem 2.2.2. *There is a rational representation $G \to \mathrm{GL}(V)$ and a one dimensional subspace $L \subset V$ such that for any scheme S,*

$$H(S) = \{g \in G(S) \mid g \cdot (L \otimes \mathcal{O}_S) \subset L \otimes \mathcal{O}_S\}.$$

Proof. It suffices to consider only affine $S = \operatorname{Spec} A$. As previously noted, I is finitely generated. The proof of [Hum75, Theorem 8.7] implies there exists a finite dimensional subspace $V \subset \mathcal{O}_G$ containing finite generating sets of I as an ideal and $\mathcal{O}_G$ as a k-algebra. Furthermore, V is stable under ρ and ρ induces a faithful rational representation $G \to \mathrm{GL}(V)$. Then we can take $W := V \cap I$ a finite subspace of V, which generates I. It easily follows from Lemma 2.2.1 that

$$H(S) = \{g \in G(S) \mid g \cdot (W \otimes \mathcal{O}_S) \subset W \otimes \mathcal{O}_S\}.$$

It remains to compress W into a line.

Let $r = \dim W$. Recall from algebraic geometry that the Grassmannian $\mathrm{Gr}(r, V)$ is a scheme satisfying

$$\mathrm{Hom}(S, \mathrm{Gr}(r, V)) = \{\mathcal{M} \text{ rank } r \text{ vector bundle}, \mathcal{M} \hookrightarrow V \underset{k}{\otimes} \mathcal{O}_S\}$$

where the cokernel of $\mathcal{M} \hookrightarrow V \otimes \mathcal{O}_S$ must also be a vector bundle. Hence, W defines a k-point of $\mathrm{Gr}(r, V)$. The Plücker embedding

$$\mathrm{Gr}(r, V) \hookrightarrow \mathbf{P}(\textstyle\bigwedge^r V)$$

is a closed embedding sending $\mathcal{M} \hookrightarrow V \otimes \mathcal{O}_S$ to $\bigwedge^r \mathcal{M} \hookrightarrow \bigwedge^r V \otimes \mathcal{O}_S$. From the functor of points perspective, it is clear that the G-representation on V induces an action of G on $\mathrm{Gr}(r, V)$. Consider the map

$$G \xrightarrow{\mathrm{id} \times W} G \times \mathrm{Gr}(r, V) \to \mathrm{Gr}(r, V).$$

In the previous paragraph we have proved that the left square in the following diagram is Cartesian.

$$\begin{array}{ccccc} H & \longrightarrow & k & \xrightarrow{\mathrm{id}} & k \\ \downarrow & & \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle \bigwedge^r W} \\ G & \longrightarrow & \mathrm{Gr}(r, V) & \hookrightarrow & \mathbf{P}(\bigwedge^r V) \end{array}$$

The right square is also Cartesian since the Plücker embedding is closed, k is a field, and we already know the square commutes. Thus the big square is Cartesian, which implies that if we consider the representation $G \to \mathrm{GL}(\bigwedge^r V)$, we have

$$H(S) = \{g \in G(S) \mid g \cdot (\textstyle\bigwedge^r W \otimes \mathcal{O}_S) \subset \bigwedge^r W \otimes \mathcal{O}_S\}.$$

Since $\bigwedge^r W$ is one dimensional, we are done. □

2.3. **Stabilizers.** The proof of Theorem 2.2.2 hints at a functorial notion of a *stabilizer* or *isotropy group*. Let X be a scheme with G-action, and $x \in X(k)$. Then we can define the G-equivariant map $G \to G \times X \to X : g \mapsto g.x$. The closed subgroup $\mathrm{Stab}_G(x)$ is defined as the fibered product

$$\begin{array}{ccc} \mathrm{Stab}_G(x) & \longrightarrow & k \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ G & \longrightarrow & X \end{array}$$

which has a group structure via functor of points.

In the proof of Theorem 2.2.2, we have shown the following:

Corollary 2.3.1. *There is a rational G-representation V and $x \in \mathbf{P}(V)(k)$ such that $H \simeq \mathrm{Stab}_G(x)$.* □

We will use but omit the proof of the following lemma, which requires some extra technical arguments to deal with the case when G is non-reduced.

Lemma 2.3.2 ([DG70, III, §3, Proposition 5.2])**.** *Let X be a scheme of finite type with a G-action. For $x \in X(k)$, the fppf sheaf $G/\mathrm{Stab}_G(x)$ is representable, and the canonical morphism $G/\mathrm{Stab}_G(x) \hookrightarrow X$ is a locally closed embedding.* □

2.4. **Proof of Theorem 2.1.1.**

Proof. Corollary 2.3.1 and Lemma 2.3.2 together tell us that G/H is representable by a quasi-projective scheme, and we have $G \to G/H \hookrightarrow \mathbf{P}(V)$ via G-action on $x \in \mathbf{P}(V)(k)$. The pullback of $\mathcal{O}(1)$ on $\mathbf{P}(V)$ to G/H is by definition a very ample line bundle. To show G-equivariance of the pullback, it is enough to show $\mathcal{O}(1)$ is a G-equivariant line bundle on $\mathbf{P}(V)$. For arbitrary scheme S, take $s \in \mathbf{P}(V)(S)$ and $g \in G(S)$. We want isomorphisms $s^*\mathcal{O}(1) \simeq (g.s)^*\mathcal{O}(1)$ satisfying the natural compatibilities. The S-point s is the same as a surjection $V \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{L}$, where $\mathcal{L} = s^*\mathcal{O}(1)$. The action of g on s changes the surjection, but leaves $\mathcal{L}$ the same, so we have $\mathcal{L} = (g.s)^*\mathcal{O}(1)$. Therefore our isomorphism is just identity, which shows $\mathcal{O}(1)$ is G-equivariant.

The map $G \to G/H$ is a categorical quotient since it satisfies the universal property in the category of fppf sheaves, and the Yoneda embedding of schemes is fully faithful. Now that we know G/H is a scheme, we can consider the identity map $\mathrm{id}_{G/H}$. Since $G \to G/H$ is an epimorphism of fppf sheaves, there exists fppf covering $U \to G/H$ that factors through G. From the remark in 2.1, we see that we have Cartesian squares

$$\begin{array}{ccccc} U \times H & \longrightarrow & G \times H & \xrightarrow{\text{mult}} & G \\ \big\downarrow & & \big\downarrow{\scriptstyle p_2} & & \big\downarrow \\ U & \longrightarrow & G & \longrightarrow & G/H \end{array}$$

Clearly H/k is fppf, which by base change implies $U \times H \to U$ is also fppf. The map $U \to G/H$ is a fppf covering, and the property of being fppf is fppf local, so we deduce that $G \to G/H$ is fppf. This shows 1.1.2(1) is satisfied. Condition 1.1.2(2) is satisfied since fppf maps are open and surjective. □

Note that $G \to G/H$ fppf and $G \times H \simeq G \times_{G/H} G$ implies G is an H-bundle in the fppf topology over G/H.

3. Smooth case

For this section, assume H, G are smooth over k. We note that every algebraic group over a field of characteristic zero is smooth [DG70, II, §6, n°1, Théorème de Cartier]. An algebraically closed field is perfect, so by generic smoothness and homogeneity, smoothness and geometric reducedness of an algebraic group are equivalent.

In this situation, the quotient G/H satisfies stronger conditions.

Theorem 3.0.1 ([Hum75, §12], [Spr09, Theorem 5.5.5])**.** *Let $H \subset G$ be as stated. The map $G \to G/H$ is a geometric quotient, and G/H is smooth over k.*

Proof. From the sheaf perspective, $(G \times \bar{k})/(H \times \bar{k}) \simeq (G/H) \times \bar{k}$. The proof of Theorem 2.1.1 then shows that if H, G are geometrically reduced, so is G/H. Homogeneity implies G/H is smooth. We also note that $G \to G/H$ being an H-bundle implies it is a smooth map.

By Theorem 2.1.1, we only need to check 1.1.2(3). Let G° denote the connected component of G, and $H' = G^\circ \cap H$. Then G/H is a disjoint union of G°/H', so we can reduce to the case when G is connected, hence irreducible. Let $\pi : G \to G/H$, take open $U \subset G/H$, and set $V := \pi^{-1}(U)$. Consider some section $f : V \to \mathbf{A}^1$ that is H-invariant. As V is an open inside affine G, it is separated over k. Thus f is separated, and the graph $\Gamma(f) : V \to V \times \mathbf{A}^1$ is a closed embedding. Let $\Gamma' \subset U \times \mathbf{A}^1$ denote the image of the projection $\pi' : V \times \mathbf{A}^1 \to U \times \mathbf{A}^1$. Since π' is fppf, hence open, H-invariance of f implies

$$\pi'(V \times \mathbf{A}^1 - \Gamma(f)) = U \times \mathbf{A}^1 - \Gamma'.$$

Thus Γ' is closed, and we can give it the reduced scheme structure, so we have commutative diagram

$$\begin{array}{ccc} V & \xhookrightarrow{\Gamma(f)} & V \times \mathbf{A}^1 \\ \downarrow & & \downarrow \\ \Gamma' & \hookrightarrow & U \times \mathbf{A}^1 \end{array}$$

Since G is integral, we have V, Γ', U are also integral. Consider

$$\phi : \Gamma' \hookrightarrow U \times \mathbf{A}^1 \xrightarrow{p_1} U,$$

which by construction is a bijection on sets. In fact, the same argument applied after a base change by $\bar{k}/k$ shows that $\phi \times_k \bar{k}$ is also a bijection on sets. The map $V \to \Gamma' \to U$ is just $\pi|_V$, which is smooth. Thus $\operatorname{Spec}\operatorname{Frac}(V) \to \operatorname{Spec}\operatorname{Frac}(U)$ is smooth, so we have field extensions

$$\operatorname{Frac}(U) \subset \operatorname{Frac}(\Gamma') \subset \operatorname{Frac}(V)$$

and $\operatorname{Frac}(U) \subset \operatorname{Frac}(V)$ is separably generated. By [Eis95, Corollary A1.6], this implies $\operatorname{Frac}(U) \subset \operatorname{Frac}(\Gamma')$ is separable. Now $\phi \times_k \bar{k}$ bijection and [GD60, 6.4.7] implies $[\operatorname{Frac}(\Gamma') : \operatorname{Frac}(U)] = 0$, i.e., ϕ is birational. Then ϕ is a bijective, birational map of integral schemes of finite type. Moreover, U is smooth over k, and hence normal. Zariski's Main Theorem implies ϕ is an isomorphism. This gives a section $U \simeq \Gamma' \to U \times \mathbf{A}^1 \to \mathbf{A}^1$. We have shown $G \to G/H$ satisfies 1.1.2(3), and hence is a geometric quotient. $\square$

4. Normal subgroups

We mention the following result but omit the proof.

Theorem 4.0.2 ([Wat79, Theorem 16.3])**.** *Let G be an affine group scheme over a field k. Let N be a closed normal subgroup. Then the fppf sheaf G/N is representable by an affine group scheme over k.*

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