HIGHER DIRECT IMAGES OF COHERENT SHEAVES UNDER A PROPER MORPHISM

JONATHAN WANG

Contents

The aim of this presentation is to prove the following theorem.

Theorem 1. *Let S be a Noetherian scheme and $\pi : X \to S$ a proper morphism. For $\mathcal{F} \in \mathrm{Coh}(X)$, the higher direct images $R^i\pi_*(\mathcal{F})$ are all coherent sheaves, for $i \geq 0$.*

We will only need to consider Noetherian schemes. Note that if X is a Noetherian scheme, then it is topologically Noetherian, so any open subset is quasi-compact. It follows that for any morphism $f : X \to Y$, the direct image f_* sends q.c. sheaves on X to q.c. sheaves on Y. After some preliminary facts, we define the higher direct image functors $R^if_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ in the Noetherian case.

1. Preliminaries

1.1. **Injective resolutions.** Let $\mathcal{C}$ be an abelian category. An object $I \in \mathcal{C}$ is *injective* if the functor $\mathrm{Hom}(-, I)$ is exact. An injective resolution of an object $A \in \mathcal{C}$ is an exact sequence

$$0 \to A \to I^0 \to I^1 \to \dots$$

where $I^\bullet$ are injective. We say $\mathcal{C}$ has enough injectives if every object has an injective resolution. It is easy to see that this is equivalent to saying every object can be embedded in an injective object. The following lemma describes the relation between maps of objects and maps of injective resolutions.

Lemma 2. *Let $\phi : A \to B$ be a map, and let $I^\bullet, J^\bullet$ be injective resolutions of A, B respectively. Then there exists a map of complexes $\varphi^\bullet : I^\bullet \to J^\bullet$ making the following diagram commute.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & I^0 & \xrightarrow{d_A^0} & I^1 & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \varphi^0} & & \downarrow{\scriptstyle \varphi^1} & & \\ 0 & \longrightarrow & B & \xrightarrow[j]{} & J^0 & \xrightarrow[d_B^0]{} & J^1 & \longrightarrow & \dots \end{array}$$

Moreover, if there are two maps $\varphi^\bullet, \psi^\bullet$ satisfying the conditions, then $\varphi^\bullet, \psi^\bullet$ are homotopic.

Proof. Since J^0 is injective, there exists a map $\varphi^0 : I^0 \to J^0$ such that $j \circ \phi = \varphi^0 \circ i$. Next since $d_B^0 \circ \varphi^0 \circ i = d_B^0 \circ j \circ \phi = 0$, the map $d_B^0 \circ \varphi^0$ factors to $I^0/A \simeq \mathrm{Im}(d_A^0) \to J^1$. Since $0 \to \mathrm{Im}(d_A^0) \to I^1$ and J^1 is injective, there exists map $\varphi^1 : I^1 \to J^1$ such that $d_B^0 \circ \varphi^0 = \varphi^1 \circ d_A^0$. We can now define φ^i for $i > 1$ by induction, using the same method.

Suppose both $\varphi^\bullet, \psi^\bullet$ induce the map ϕ. We construct a homotopy $h^\bullet$ from $\varphi^\bullet$ to $\psi^\bullet$. It is understood that $h^0 = 0$. Since $\varphi^0 \circ i = \psi^0 \circ i = j \circ \phi$, we have that $\varphi^0 - \psi^0$ factors to $I^0/A \to J^0$. Using injectivity of J^0 and the same argument as before, there exists a map $h^1 : I^1 \to J^0$ such that $\varphi^0 - \psi^0 = h^1 \circ d_A^0$. We now construct h^{i+1} for $i > 0$ inductively. Suppose that $h^j : I^j \to J^{j-1}$ exist for $j \leq i$ and satisfy $\varphi^j - \psi^j = h^{j+1} \circ d_A^j + d_B^{j-1} \circ h^j$ for $j < i$. Then since $\varphi^\bullet, \psi^\bullet$ are maps of complexes,

$$(-d_B^{i-1} \circ h^i + \varphi^i - \psi^i) d_A^{i-1} = d_B^{i-1}(-h^i \circ d_A^{i-1} + \varphi^{i-1} - \psi^{i-1}) = d_B^{i-1} \circ d_B^{i-2} \circ h^{i-1} = 0.$$

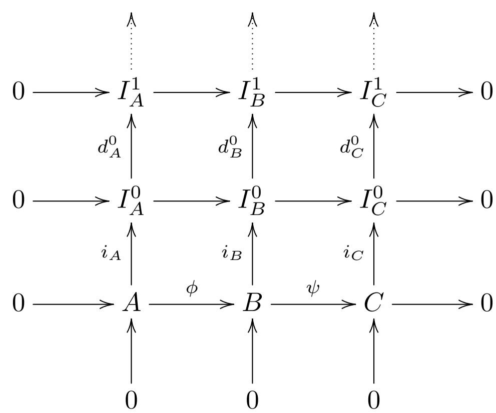
Using injectivity of J^i, there exists $h^{i+1} : I^{i+1} \to J^i$ such that

$$h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i = \varphi^i - \psi^i.$$

This completes the inductive step, so we have a homotopy $h^\bullet$ from $\varphi^\bullet$ to $\psi^\bullet$. □

We next show that with enough injectives, short exact sequences of objects in $\mathcal{C}$ give rise to short exact sequences of injective resolutions.

Lemma 3. *Let $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ be a short exact sequence in $\mathcal{C}$. Suppose A, C have injective resolutions $I_A^\bullet, I_C^\bullet$. Then there exists an injective resolution $I_B^\bullet$ of B and a split short exact sequence of complexes $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ such* *that the following diagram commutes.*



Proof. See [Wei94, Horseshoe Lemma 2.2.8]. □

1.2. **Derived functors.** Let $\mathcal{C}$ have enough injectives. Let $F : \mathcal{C} \to \mathcal{D}$ be a left exact, additive functor. We define the *right derived functors* R^iF of F for $i \geq 0$ as follows. For $A \in \mathcal{C}$, take an injective resolution $I^\bullet$ of A. Apply F to get the chain complex

$$0 \to F(I^0) \to F(I^1) \to \dots$$

and set $R^iF(A) := H^i(F(I^\bullet))$ to be the i-th cohomology of the complex. Since F is left exact, $R^0F(A) \simeq F(A)$.

Lemma 4. *For $A \in \mathcal{C}$, let $I^\bullet, J^\bullet$ be two injective resolutions of A. There is a canonical isomorphism $R^iF_I(A) \simeq R^iF_J(A)$ for $i \geq 0$.*

Proof. By Lemma 2, we can lift the identity map id_A to maps of complexes $\varphi^\bullet : I^\bullet \to J^\bullet$ and $\psi^\bullet : J^\bullet \to I^\bullet$. The composition $\psi \circ \varphi$ induces id_A, so by the second part of Lemma 2, we have that $\psi \circ \varphi$ is homotopic to id_I. Similarly $\varphi \circ \psi$ is homotopic to id_J. Thus φ is a homotopy equivalence, and is determined by id_A up to homotopy. Since F is additive, $F(\varphi^\bullet) : F(I^\bullet) \to F(J^\bullet)$ defines a homotopy equivalence. Thus $R^iF_I(A) \simeq R^iF_J(A)$, and since φ is determined up to homotopy, the isomorphism is canonical. □

Thus R^iF is defined up to isomorphism independent of choice of injective resolution. Note that if $I \in \mathcal{C}$ is injective, then $0 \to I \to I \to 0$ is an injective resolution, so $R^iF(I) = 0$ for $i > 0$.

1.2.1. *Long exact sequences.* We show that a short exact sequence of objects in $\mathcal{C}$ give a long exact sequence of a derived functor.

Lemma 5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{C}$. Then there exists a long exact sequence*

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots$$

Moreover, the long exact sequences are natural: if we have a commutative diagram

$$\begin{CD} 0 @>>> A @>>> B @>>> C @>>> 0 \\ @. @VVV @VVV @VVV @. \\ 0 @>>> A' @>>> B' @>>> C' @>>> 0 \end{CD}$$

then the corresponding long exact sequences commute. That is, we have

$$\begin{CD} \dots @>>> R^iF(A) @>>> R^iF(B) @>>> R^iF(C) @>>> \dots \\ @. @VVV @VVV @VVV @. \\ \dots @>>> R^iF(A') @>>> R^iF(B') @>>> R^iF(C') @>>> \dots \end{CD}$$

Proof. Since $\mathcal{C}$ has enough injectives, take injective resolutions $I_A^\bullet, I_C^\bullet$ of A, C. From Lemma 3, there exists an injective resolution $I_B^\bullet$ of B such that

$$0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$$

is a split short exact sequence of complexes, i.e., $I_B^i \simeq I_A^i \oplus I_C^i$. Since F is additive, we have $F(I_A^i \oplus I_C^i) \simeq F(I_A^i) \oplus F(I_C^i)$ under the canonical maps (PS 2, Problem 1). Hence $0 \to F(I_A^\bullet) \to F(I_B^\bullet) \to F(I_C^\bullet) \to 0$ is also a split exact sequence. Short exact sequence of complexes gives a long exact sequence of cohomology:

$$0 \to H^0(F(I_A^\bullet)) \to H^0(F(I_B^\bullet)) \to H^0(F(I_C^\bullet)) \to H^1(F(I_A^\bullet)) \to \dots$$

By definition of R^iF, this gives the desired long exact sequence. □

2. Higher direct images

Let X be a scheme and $Sh^{\mathcal{O}_X}(X)$ the category of $\mathcal{O}_X$-modules.

Lemma 6. *The category $Sh^{\mathcal{O}_X}(X)$ has enough injectives.*

Proof. Take $\mathcal{F} \in Sh^{\mathcal{O}_X}(X)$. For each point $x \in X$, we have $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module. Since the category of modules of rings has enough injectives (Math 221, PS 8, Problem 2(f)), there exists an injective $\mathcal{O}_{X,x}$-module I_x with $\mathcal{F}_x \hookrightarrow I_x$. Let $i_x : \{x\} \to X$ denote the inclusion of the point x into X. Then $i_{x,*}(I_x)$ is the skyscraper sheaf at x with stalk I_x, which is an $\mathcal{O}_X$-module. From the universal property of inductive limit, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, i_{x,*}(I_x)) \simeq \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x),$$

i.e., taking the stalk is left adjoint to $i_{x,*}$. Since taking the stalk is an exact functor[1] and I_x is injective, $i_{x,*}(I_x)$ is injective in $Sh^{\mathcal{O}_X}(X)$. Since $i_{x,*}$ is right adjoint, it is left exact, and the unit map gives $\mathcal{F} \to i_{x,*}(\mathcal{F}_x) \hookrightarrow i_{x,*}(I_x)$. A product of injectives is injective, so

$$\mathcal{I} = \prod_{x \in X} i_{x,*}(I_x) \in Sh^{\mathcal{O}_X}(X)$$

is injective. By universal property of the product, we have a map $\mathcal{F} \to \mathcal{I}$. Taking the stalk at $x \in X$, the x-th component of $\mathcal{F}_x \to \mathcal{I}_x$ corresponds to $\mathcal{F}_x \to \mathcal{F}_x \to I_x$, which is injective. Thus $\mathcal{F} \to \mathcal{I}$ is injective on stalks and hence injective as a map of sheaves. □

[1] Thanks to Kaloyan for pointing out that right adjoints do not send injectives to injectives unless the left adjoint is exact.

Let $f : X \to Y$ be a map of schemes. Then $f_* : Sh^{\mathcal{O}_X}(X) \to Sh^{\mathcal{O}_Y}(Y)$ is a right adjoint, which implies it is both additive and left exact. Thus we can define $R^i f_* : Sh^{\mathcal{O}_X}(X) \to Sh^{\mathcal{O}_Y}(Y)$ to be the right derived functors of f_*.

Lemma 7. *Take $\mathcal{I} \in Sh^{\mathcal{O}_X}(X)$ be injective. For an open $U \subset X$, $\mathcal{I}|_U$ is injective in $Sh^{\mathcal{O}_U}(U)$.*

Proof. Let $j : U \to X$ be the open embedding. By PS 5, Problem 10, the restriction functor $j^* : Sh^{\mathcal{O}_X}(X) \to Sh^{\mathcal{O}_U}(U)$ admits a left adjoint $j_! : Sh^{\mathcal{O}_U}(U) \to Sh^{\mathcal{O}_X}(X)$. Since $\mathcal{I}|_U \simeq j^*(\mathcal{I})$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_U}(-, \mathcal{I}|_U) \simeq \mathrm{Hom}_{\mathcal{O}_X}(j_!(-), \mathcal{I}).$$

By PS 7, Problem 7(c), $j_!$ is exact, so $\mathcal{I}|_U$ is injective. □

Lemma 8. *If $U \subset X$ is open, then*

$$R^i f_*(\mathcal{F})|_U \simeq R^i f'_*(\mathcal{F}|_{f^{-1}(U)})$$

where $f' : f^{-1}(U) \to U$ is the restricted map.

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$ be an injective resolution. Since restriction is exact,

$$R^i f_*(\mathcal{F})|_U = H^i(f_*(\mathcal{I}^\bullet))|_U \simeq H^i(f_*(\mathcal{I}^\bullet)|_U) \simeq H^i(f'_*(\mathcal{I}^\bullet|_{f^{-1}(U)})).$$

By Lemma 7, $0 \to \mathcal{F}|_{f^{-1}(U)} \to \mathcal{I}^0|_{f^{-1}(U)} \to \mathcal{I}^1|_{f^{-1}(U)} \to \dots$ is an injective resolution of $\mathcal{F}|_{f^{-1}(U)}$, so $H^i(f'_*(\mathcal{I}^\bullet|_{f^{-1}(U)})) = R^i f'_*(\mathcal{F}|_{f^{-1}(U)})$. □

We now look at Noetherian schemes and quasi-coherent sheaves on them. Let $H^i(X, -) : Sh^{\mathcal{O}_X}(X) \to \Gamma(X, \mathcal{O}_X)$-mod be the right derived functor of the global section $\Gamma(X, -)$, which we call the sheaf cohomology.

We state the following lemma without proof (cf. [Har77, III.8, Proposition 8.5]).

Lemma 9. *Let $f : X \to \operatorname{Spec} A$ be a map of schemes where X is Noetherian. Then for $\mathcal{F} \in \mathrm{QCoh}(X)$, we have*

$$R^i f_*(\mathcal{F}) \simeq \mathrm{Loc}_A(H^i(X, \mathcal{F})).$$

Now consider a general map of schemes $f : X \to Y$ with X Noetherian. Take $\mathcal{F} \in \mathrm{QCoh}(X)$. Take an affine $U \subset Y$ and let $f' : f^{-1}(U) \to U$ be the restriction map. Observe that $f^{-1}(U) \subset X$ is Noetherian. By Lemma 8, $R^i f_*(\mathcal{F})|_U \simeq R^i f'_*(\mathcal{F}|_{f^{-1}(U)})$, where $\mathcal{F}|_{f^{-1}(U)} \in \mathrm{QCoh}(f^{-1}(U))$. Lemma 9 implies that $R^i f'_*(\mathcal{F}|_{f^{-1}(U)})$ is quasi-coherent on U. This is true for every open affine $U \subset Y$, and quasi-coherence is a local property, so we deduce that $R^i f_*(\mathcal{F}) \in \mathrm{QCoh}(Y)$ for all $i \geq 0$. Thus in the case when X is Noetherian, the higher direct image functor on $\mathcal{O}_X$-modules induces $R^i f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.

2.1. **Sheaf cohomology.** We see from Lemma 9 that the higher direct image functors and sheaf cohomology are closely related. We prove some results that will be necessary later on.

2.1.1. *Flasque sheaves.* A sheaf $\mathcal{F} \in Sh(X)$ on a topological space X is *flasque* if the restriction maps $\Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ are surjective for all $V \subset U \subset X$.

Lemma 10.

(1) *If* $0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathcal{F}'$ *is flasque, then* $0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'') \to 0$ *is exact for any open* $U \subset X$.
(2) *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is exact and* $\mathcal{F}', \mathcal{F}$ *are flasque, then* $\mathcal{F}''$ *is also flasque.*
(3) *For a map* $f : X \to Y$*, the direct image* $f_*(\mathcal{F})$ *of a flasque sheaf is flasque.*

Proof. (1) We know $\Gamma(U, -)$ is left exact, so it suffices to show surjectivity. Let $g \in \Gamma(U, \mathcal{F}'')$. By surjectivity of $\mathcal{F} \to \mathcal{F}''$, there exists a cover $U = \bigcup U_i$ and $f_i \in \Gamma(U_i, \mathcal{F})$ such that $\psi(f_i) = g|_{U_i}$. Consider the set of all pairs (V, f) such that $V \subset U$, $f \in \Gamma(V, \mathcal{F})$, and $\psi(f) = g|_V$. We can partially order $\{(V, f)\}$ by $(V_1, f_1) \le (V_2, f_2)$ if $V_1 \subset V_2$ and $f_2|_{U_1} = f_1$. Given an increasing chain $\{(V_i, f_i)\}$, by the sheaf axiom on $\mathcal{F}, \mathcal{F}''$ there exists $f \in \Gamma(\bigcup V_i, \mathcal{F})$ such that $(\bigcup V_i, f)$ is maximal with $\psi(f) = g|_{\bigcup V_i}$. Now by Zorn's Lemma, there exists a maximal element (V, f). Suppose $V \neq U$. Then since U_i cover U, there exists $U_i \not\subset V$ with $\psi(f_i) = g|_{U_i}$. Let $V' = V \cap U_i$ and $U' = V \cup U_i$. Observe that $\psi(f|_{V'} - f_i|_{V'}) = 0$, so $f|_{V'} - f_i|_{V'} = \phi(h')$ for $h' \in \Gamma(V', \mathcal{F}')$ by left exactness of $\Gamma(V', -)$. Since $\mathcal{F}'$ is flasque, there exists $h \in \Gamma(U_i, \mathcal{F}')$ such that $h|_{V'} = h'$. Now f and $f_i + \phi(h)$ agree on the intersection, so they can be glued to get $f' \in \Gamma(U', \mathcal{F})$. By uniqueness of the sheaf axiom, $\psi(f') = g|_{U'}$. This contradicts maximality of V, so we must have $V = U$. Therefore $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$ is surjective.

(2) For $V \subset U \subset X$, from (1) we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(U, \mathcal{F}') & \longrightarrow & \Gamma(U, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(V, \mathcal{F}') & \longrightarrow & \Gamma(V, \mathcal{F}) & \longrightarrow & \Gamma(V, \mathcal{F}'') & \longrightarrow & 0 \end{array}$$

This implies $\Gamma(U, \mathcal{F}'') \to \Gamma(V, \mathcal{F}'')$ is surjective, so $\mathcal{F}''$ is flasque.

(3) This is easy to see from the definition of f_*. □

Lemma 11. *An injective* $\mathcal{O}_X$*-module* $\mathcal{I}$ *is flasque.*

Proof. For open subsets $V \subset U \subset X$, let $j : V \hookrightarrow X$ and $j' : U \hookrightarrow X$. Then we have an inclusion $0 \to j_!(\mathcal{O}_V) \to j'_!(\mathcal{O}_U)$ of $\mathcal{O}_X$-modules: injectivity is clear on presheaves, and sheafification is exact. Since $\mathcal{I}$ is injective, this implies $\mathrm{Hom}(j'_!(\mathcal{O}_U), \mathcal{I}) \to \mathrm{Hom}(j_!(\mathcal{O}_V), \mathcal{I}) \to 0$. By adjunction, $\mathrm{Hom}(j'_!(\mathcal{O}_U), \mathcal{I}) \simeq \mathrm{Hom}(\mathcal{O}_U, \mathcal{I}|_U) \simeq \Gamma(U, \mathcal{I})$. Therefore $\Gamma(U, \mathcal{I}) \to \Gamma(V, \mathcal{I}) \to 0$. □

Lemma 12. *If* $\mathcal{J} \in Sh^{\mathcal{O}_X}(X)$ *is flasque, then* $H^i(X, \mathcal{J}) = 0$ *for* $i > 0$.

Proof. Since $Sh^{\mathcal{O}_X}(X)$ has enough injectives, we can embed $\mathcal{J} \to \mathcal{I}$ for $\mathcal{I}$ injective. Taking cokernel, we have the short exact sequence

$$0 \to \mathcal{J} \to \mathcal{I} \to \mathcal{F} \to 0.$$

By Lemma 10.1, we have the exact sequence

$$0 \to \Gamma(X, \mathcal{J}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{F}) \to 0.$$

Taking the long exact sequence, this implies $H^1(X, \mathcal{J}) = 0$. This is true for all flasque sheaves. Now for $i > 1$, we have from the long exact sequence

$$H^{i-1}(X, \mathcal{I}) \to H^{i-1}(X, \mathcal{F}) \to H^i(X, \mathcal{J}) \to H^i(X, \mathcal{I}).$$

By Lemma 10.2, $\mathcal{F}$ is also flasque. Since $\mathcal{I}$ is injective, $H^{i-1}(X, \mathcal{I}) = 0$ and $H^i(X, \mathcal{I}) = 0$. Therefore $H^{i-1}(X, \mathcal{F}) \simeq H^i(X, \mathcal{J})$, which we conclude is equal to 0 by induction on i. □

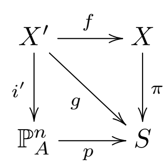
We state the following lemma without proof (cf. [Har77, III.4, Theorem 4.5]).

Lemma 13. *Let X be a Noetherian separated scheme. Then for $\mathcal{F} \in \mathrm{QCoh}(X)$, there is a natural isomorphism $\check{H}^i(X, \mathcal{F}) \simeq H^i(X, \mathcal{F})$.*

3. The main argument

We now prove Theorem 1. Coherence is a local property, so by Lemma 8, we can assume $S = \operatorname{Spec} A$ is affine. Since proper morphisms are of finite type, X is also Noetherian. Closed embeddings are proper, so by Noetherian induction, we can assume that for any proper closed subscheme $i : Y \to X$, the theorem holds for $\pi \circ i$.

Now by Chow's Lemma, there exists a scheme X' equipped with map $f : X' \to X$ such that $g = \pi \circ f : X' \to S$ is projective, and there exists a dense open subscheme $U \subset X$ such that the map $f^{-1}(U) \to U$ is an isomorphism. By definition of a projective morphism, we have the commutative diagram



Closed embeddings are proper, and we showed in class that $p : \mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. By composition, $g = p \circ i'$ is proper, which implies X' is also Noetherian. Let $\mathcal{F} \in \mathrm{Coh}(X)$ and $\mathcal{F}' := f^*\mathcal{F}$. If M is a f.g. A-module, then $B \otimes_A M$ is a f.g. B-module; this implies pullback preserves coherence, so $\mathcal{F}' \in \mathrm{Coh}(X')$. Additionally, $f_*(\mathcal{F}') \in \mathrm{QCoh}(X)$. Consider the unit map $\mathcal{F} \to f_*(\mathcal{F}')$. Let $f' : f^{-1}(U) \to U$ denote the restriction map. Then

$$f_*(\mathcal{F}')|_U \simeq f'_*(\mathcal{F}'|_{f^{-1}(U)}) \simeq f'_* f'^*(\mathcal{F}),$$

so the restriction to U of $\mathcal{F}|_U \to f_*(\mathcal{F}')|_U$ is the unit of f'^*, f'_*. Since f' is an isomorphism, the unit is also an isomorphism. Take the exact sequence

$$0 \to K_1 \to \mathcal{F} \to f_*(\mathcal{F}') \to C \to 0.$$

We have shown that $K_1|_U = 0$ and $C|_U = 0$. Since X is Noetherian, subsheaves and quotients of coherent sheaves are both coherent. Therefore $K_1, C \in \mathrm{Coh}(X)$.

Lemma 14. *Let X be a scheme and $\mathcal{F} \in \mathrm{Coh}(X)$ such that $\mathcal{F}|_U = 0$ for a nonempty open $U \subset X$. Then there exists a proper closed subscheme $i : Y \to X$ with $\mathcal{F}$ in the essential image of $i_* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$.*

Proof. We define the ideal sheaf $\mathrm{ann}(\mathcal{F})$ by

$$\Gamma(\operatorname{Spec} B, \mathrm{ann}(\mathcal{F})) = \mathrm{ann}_B(\Gamma(\operatorname{Spec} B, \mathcal{F})),$$

where $\operatorname{ann}_B(M) = \{b \in B \mid bm = 0 \ \forall m \in M\}$ for a B-module M. Let M be f.g. with generators $m_1, \dots, m_k$. Then for $f \in B$, $a/f^j \in \operatorname{ann}(M_f)$ implies $f^{n_i} a \cdot m_i = 0$ for all $i = 1, \dots, k$. Letting $N = \max(n_i)$, we have $f^N a \in \operatorname{ann}(M)$, so $a/f^j = f^N a/f^{N+j} \in \operatorname{ann}(M)_f$. The other direction is obvious, so $\operatorname{ann}(M)_f = \operatorname{ann}(M_f)$. Hence $\operatorname{ann}(\mathcal{F})$ is a well-defined q.c. subsheaf of $\mathcal{O}_X$. Let Y be the corresponding closed subscheme. We claim that $i_* i^*(\mathcal{F}) \simeq \mathcal{F}$. Locally, this says that $B/\operatorname{ann}(M) \otimes_B M \simeq M$ as B-modules, which is true since M clearly has the structure of a $B/\operatorname{ann}(M)$-module.

To see that Y is proper, let $\operatorname{Spec} B \cap U \neq \emptyset$. Then the intersection contains U_f for $f \in B$ non-nilpotent. As $\mathcal{F}|_U = 0$, we have $B_f \otimes_B M = 0$ for $M = \Gamma(\operatorname{Spec} B, \mathcal{F})$. Since M is f.g., the same argument as before implies $f \in \operatorname{ann}(M)$. Therefore $V(\operatorname{ann}(M)) \subset \operatorname{Spec} B$ is proper, so $Y \subset X$ is proper. □

It follows from Lemma 14 and Noetherian induction that Theorem 1 is true for K_1, C. Suppose it is also true for $f_*(\mathcal{F}')$. Then we factor $\mathcal{F} \to f_*(\mathcal{F}')$ to $\mathcal{F} \to K_2 \to f_*(\mathcal{F}')$ such that

$$0 \to K_1 \to \mathcal{F} \to K_2 \to 0 \qquad 0 \to K_2 \to f_*(\mathcal{F}') \to C \to 0$$

are exact.

Lemma 15. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in* $\operatorname{Coh}(X)$*. If any two sheaves in the sequence satisfy Theorem 1, then the third does as well.*

Proof. Taking the long exact sequence, we have exact sequence

$$R^{i-1}\pi_*(C) \to R^i\pi_*(A) \to R^i\pi_*(B) \to R^i\pi_*(C) \to R^{i+1}\pi_*(A).$$

As an example, suppose A, C satisfy Theorem 1. Then $R^i\pi_*(B)$ is sandwiched between two coherent sheaves. Subsheaves and quotients of coherent sheaves are coherent, so by extension, $R^i\pi_*(B)$ must also be coherent (f.g. modules are stable under extension). The other cases are similar. □

Assuming that $f_*(\mathcal{F}')$ satisfies Theorem 1, applying Lemma 15 to the second sequence implies that K_2 also satisfies. Applying the lemma again, we conclude that $R^i\pi_*(\mathcal{F}) \in \operatorname{Coh}(S)$ for all $i \geq 0$. Thus it remains to show that $R^i\pi_*(f_*(\mathcal{F}'))$ is always coherent on S.

Lemma 16. *Let $g : X' \to S$ be a projective morphism with S Noetherian. Then for $\mathcal{F}' \in \operatorname{Coh}(X')$, we have $R^i g_*(\mathcal{F}')$ coherent for $i \geq 0$.*

Proof. As before, we can assume $S = \operatorname{Spec} A$ is affine. By Lemma 9, we know that $R^i g_*(\mathcal{F}') \simeq \operatorname{Loc}_A(H^i(X', \mathcal{F}'))$. Therefore $R^i g_*(\mathcal{F}')$ being coherent is equivalent to $H^i(X', \mathcal{F}')$ being f.g. as an A-module. The map g is separated and $\operatorname{Spec} A$ is separated, so X' is also separated. Thus Čech cohomology agrees with sheaf cohomology by Lemma 13. Closed embeddings preserve coherence, so $i_*(\mathcal{F}') \in \operatorname{Coh}(\mathbb{P}^n_A)$. Then $\check{H}^i(X', \mathcal{F}') \simeq \check{H}^i(\mathbb{P}^n_A, i_*(\mathcal{F}'))$, which is a f.g. A-module by Serre's Theorem. □

Lemma 16 shows that $R^i(\pi \circ f)_*(\mathcal{F}') \in \operatorname{Coh}(S)$, while we would like $R^i\pi_*(f_*(\mathcal{F}')) \in \operatorname{Coh}(S)$. To show the relation between these two sheaves, we must use spectral sequences.

4. Spectral sequences

Let $\mathcal{C}$ be an abelian category. A *cohomology spectral sequence* in $\mathcal{C}$ starting at page a is a collection of objects $\{E_r^{p,q}\}$ for $p, q \in \mathbb{Z}$ and $r \geq a$, together with the datum of: differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} d_r^{p,q} = 0$, and isomorphisms $\ker d_r^{p,q} / \operatorname{Im} d^{p-r,q+r-1} \xrightarrow{\sim} E_{r+1}^{p,q}$.

We state the following result on existence of spectral sequences without proof (the proof is very technical, cf. [Wei94, 5.6]).

Lemma 17. *Let $C^{\bullet,\bullet}$ be a first quadrant bicomplex in $\mathcal{C}$. Then there are two canonical spectral sequences ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ both starting at page 0. Furthermore,*

$$ {}'E_2^{p,q} \simeq H^p(H_v^{\bullet,q}(C)) \qquad {}''E_2^{p,q} \simeq H^p(H_h^{q,\bullet}(C)). $$

Both spectral sequences ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ converge to $H^{p+q}(\operatorname{Tot}(C))$.

For bounded spectral sequences, we have that for fixed p, q the terms $E_r^{p,q} = E_\infty^{p,q}$ stabilize for all $r \gg 0$ [Wei94, Bounded Convergence 5.2.5]. We say a spectral sequence $E_r^{p,q} \Rightarrow H^{p+q}$ converges if we have $H^n \in \mathcal{C}$ each with a finite filtration

$$ 0 = F^t H^n \subset \cdots F^{p+1} H^n \subset F^p H^n \cdots \subset F^s H^n = H^n $$

so that $E_\infty^{p,q} \simeq F^p H^{p+q} / F^{p+1} H^{p+q}$.

Theorem 18 (Grothendieck spectral sequence)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be additive, left exact functors between abelian categories where $\mathcal{A}, \mathcal{B}$ have enough injectives. Suppose that F sends injectives to G-acyclics. Then for any $A \in \mathcal{A}$, there exists a first quadrant spectral sequence $E_r^{p,q}$ such that*

$$ E_2^{p,q} \simeq R^p G(R^q F(A)) \Rightarrow R^{p+q}(GF)(A). $$

We say that $B \in \mathcal{B}$ is G-acyclic if $R^i G(B) = 0$ for all $i > 0$.

Proof. Take $A \in \mathcal{A}$ and pick an injective resolution $0 \to A \to J^\bullet$. We can factor the complex $F(J^\bullet)$ into

$$ 0 \to K^0 \to F(J^0) \to C^0 \to K^1 \to F(J^1) \to \dots $$
$$ \to C^{i-1} \to K^i \to F(J^i) \to C^i \to K^{i+1} \to F(J^{i+1}) \to \dots $$

where the sequences

$$ 0 \to K^i \to F(J^i) \to C^i \to 0 \qquad 0 \to C^{i-1} \to K^i \to H^i(F(J^\bullet)) \to 0 $$

are exact. For all $i \geq 0$, pick injective resolutions $0 \to C^i \to I_C^{i,\bullet}$ and $0 \to H^i(F(J^\bullet)) \to I_H^{i,\bullet}$. By Lemma 3, there exists injective resolution $0 \to K^i \to I_K^{i,\bullet}$ such that

$$ 0 \to I_C^{i-1,\bullet} \to I_K^{i,\bullet} \to I_H^{i,\bullet} \to 0 $$

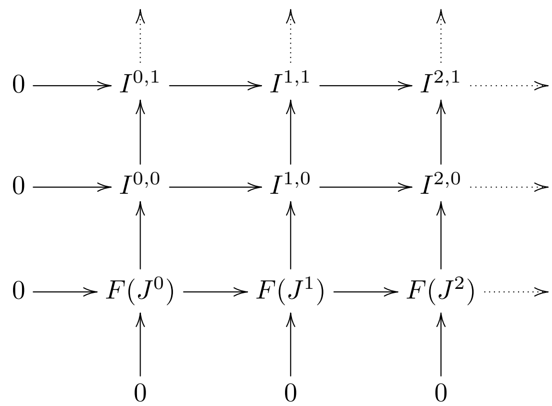
is a split short exact sequence. Applying Lemma 3 again, we get an injective resolution $0 \to F(J^i) \to I^{i,\bullet}$ such that

$$ 0 \to I_K^{i,\bullet} \to I^{i,\bullet} \to I_C^{i,\bullet} \to 0 $$

is split short exact. Therefore we have

$$ \begin{CD} I_C^{i-1,\bullet} @>>> I_K^{i,\bullet} @>>> I^{i,\bullet} @>>> I_C^{i,\bullet} \\ @AAA @AAA @AAA @AAA \\ C^{i-1} @>>> K^i @>>> F(J^i) @>>> C^i \end{CD} $$

Putting the resolutions $I^{i,\bullet}$ together for all $i \geq 0$ and connecting maps, we have the bicomplex $I^{\bullet,\bullet}$.



The choice of resolutions implies that $0 \to H^i(F(J^\bullet)) \to H_h^{i,\bullet}(I)$ is an injective resolution. Since G is additive, applying G to the split short exact sequences $0 \to I_C^{i-1,\bullet} \to I_K^{i,\bullet} \to I_H^{i,\bullet} \to 0$ and $0 \to I_K^{i,\bullet} \to I^{i,\bullet} \to I_C^{i,\bullet} \to 0$ give short exact sequences. Then

$$0 \to G(I_C^{i-1,\bullet}) \to G(I_K^{i,\bullet}) \to G(I_H^{i,\bullet}) \to 0$$
$$0 \to G(I_K^{i,\bullet}) \to G(I^{i,\bullet}) \to G(I_C^{i,\bullet}) \to 0$$

together imply there is an isomorphism $H_h^{i,\bullet}(GI) \simeq G(H_h^{i,\bullet}(I))$.

Applying Lemma 17 to the bicomplex $G(I^{\bullet,\bullet})$, we get two convergent spectral sequences

$$'E_2^{p,q} \simeq H^p(H_v^{\bullet,q}(GI)) \Rightarrow H^{p+q}(\mathrm{Tot}(GI))$$
$$''E_2^{p,q} \simeq H^p(H_h^{q,\bullet}(GI)) \Rightarrow H^{p+q}(\mathrm{Tot}(GI))$$

Since $I^{p,\bullet}$ is an injective resolution of $F(J^p)$, we see that $(H_v^{\bullet,q}(GI))^p = R^qG(F(J^p))$. By assumption F sends injectives to G-acyclics, so $R^qG(F(J^p)) \simeq GF(J^p)$ if $q = 0$ and 0 otherwise. Since $J^\bullet$ is an injective resolution of A, we have $'E_2^{p,q} \simeq R^p(GF)(A)$ if $q = 0$ and 0 otherwise. All differentials on page 2 are clearly zero, so the spectral sequence has stabilized. We conclude that $H^p(\mathrm{Tot}(GI)) \simeq R^p(GF)(A)$.

We now consider the second spectral sequence. By our previous observations, $H_h^{q,\bullet}(GI) \simeq G(H_h^{q,\bullet}(I))$ and $H_h^{q,\bullet}(I)$ is an injective resolution of $H^q(F(J^\bullet)) \simeq R^qF(A)$. Thus $''E_2^{p,q} \simeq H^p(H_h^{q,\bullet}(I)) = R^pG(R^qF(A))$. Let the Grothendieck spectral sequence $E_r^{p,q} = {''E_r^{p,q}} \Rightarrow H^{p+q}(\mathrm{Tot}(GI)) \simeq R^{p+q}(GF)(A)$. □

Corollary 19 (Leray spectral sequence)**.** *Let $f : X' \to X$ be a map of schemes. Then for $\mathcal{F}' \in Sh^{\mathcal{O}_{X'}}(X')$ there exists a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(X, R^qf_*(\mathcal{F}')) \Rightarrow H^{p+q}(X', \mathcal{F}').$$

Proof. For $\mathcal{I} \in Sh^{\mathcal{O}_{X'}}(X')$ injective, $f_*(\mathcal{I})$ is flasque by Lemma 10. By Lemma 12, $f_*(\mathcal{I})$ is $\Gamma(X, -)$ acyclic. Since $Sh^{\mathcal{O}_{X'}}(X')$ and $Sh^{\mathcal{O}_X}(X)$ have enough injectives, we can take the Grothendieck spectral sequence $E_2^{p,q} = H^p(X, R^qf_*(\mathcal{F}')) \Rightarrow H^{p+q}(X', \mathcal{F}')$, since $\Gamma(X, f_*(-)) = \Gamma(X', -)$. □

5. Finishing the proof

We now complete the proof of Theorem 1. Let $\mathcal{F}' \in \mathrm{Coh}(X')$ as before. By Lemma 16, we know that $H^{p+q}(X', \mathcal{F}')$ is a f.g. A-module. Observe that for $q > 0$, $(R^q f_*(\mathcal{F}'))|_U \simeq R^q f'_*(\mathcal{F}'|_{f^{-1}(U)})$ by Lemma 8. Since f'_* is an isomorphism, it is exact. Thus $R^q f'_*(\mathcal{F}'|_{f^{-1}(U)}) = 0$ for $q > 0$.

Note on Chow's Lemma: we claim that f is a projective morphism. Recall that we have the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^n_S & \longrightarrow & S \end{array}$$

Therefore f factors to $X' \to X \times_S \mathbb{P}^n_S \simeq \mathbb{P}^n_X \to X$. We claim that $X' \to X \times_S \mathbb{P}^n_S$ is a closed embedding, which implies f is projective. Since i' is closed and closed embeddings are stable under base change, $X' \times_S X \simeq X' \times_{\mathbb{P}^n_S} (\mathbb{P}^n_S \times_S X) \to \mathbb{P}^n_S \times_S X$ is closed, and $X' \to \mathbb{P}^n_X$ factors through $X' \to X' \times_S X \to \mathbb{P}^n_X$. We have a Cartesian square

$$\begin{array}{ccc} X' \times_X X & \longrightarrow & X' \times_S X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_S X \end{array}$$

Since π is proper and hence separated, $X \to X \times_S X$ is a closed embedding. By base change, $X' \simeq X' \times_X X \to X' \times_S X$ is also closed. Composing, we conclude that $X' \to \mathbb{P}^n_X$ is a closed embedding. Therefore f is projective.

Now by Lemma 16, we know that $R^q f_*(\mathcal{F}') \in \mathrm{Coh}(X)$. We showed $R^q f_*(\mathcal{F}')$ has proper support for $q > 0$, so by the Noetherian inductive hypothesis,

$$R^p \pi_*(R^q f_*(\mathcal{F}')) \in \mathrm{Coh}(S)$$

for $q > 0$. By Lemma 9, this implies $H^p(X, R^q f_*(\mathcal{F}'))$ is a f.g. A-module for $q > 0$. Now we take the Leray spectral sequence (Corollary 19) $E_2^{p,q} = H^p(X, R^q f_*(\mathcal{F}'))$. The previous statement is equivalent to saying $E_2^{p,q}$ is a f.g. A-module for $q > 0$. We also know that $E_2^{p,q} \Rightarrow H^{p+q}(X', \mathcal{F}')$ is f.g. By definition of convergence this implies $E_\infty^{p,q}$ is f.g. for all $p, q \geq 0$. Since $E_{r+1}^{p,q}$ is isomorphic to a quotient of a submodule of $E_r^{p,q}$, we deduce that $E_r^{p,q}$ is f.g. for $q > 0$ and $r \geq 2$. Now for $p \geq 0$, we have $d_r^{p,0} = 0$ since our spectral sequence is in the first quadrant. Therefore by the definition of a spectral sequence, we have the exact sequence

$$E_r^{p-r,r-1} \overset{d_r^{p-r,r-1}}{\to} E_r^{p,0} \to E_{r+1}^{p,0} \to 0.$$

The first term is f.g. For $r \gg 0$, $E_r^{p,0} = E_\infty^{p,0}$ is f.g. By induction and extension, we conclude that $E_r^{p,0}$ is a f.g. A-module for all $p \geq 0$, $r \geq 2$. In particular, $E_2^{p,0} = H^p(X, f_*(\mathcal{F}'))$ is f.g. Lemma 9 implies that $R^p \pi_*(f_*(\mathcal{F}')) \in \mathrm{Coh}(S)$, which complete the proof of Theorem 1.

References

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