ON THE REDUCTIVE MONOID ASSOCIATED TO A PARABOLIC SUBGROUP

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Abstract. Let G be a connected reductive group over a perfect field k. We study a certain normal reductive monoid $\overline{M}$ associated to a parabolic k-subgroup P of G. The group of units of $\overline{M}$ is the Levi factor M of P. We show that $\overline{M}$ is a retract of the affine closure of the quasi-affine variety $G/U(P)$. Fixing a parabolic P^- opposite to P, we prove that the affine closure of $G/U(P)$ is a retract of the affine closure of the boundary degeneration $(G \times G)/(P \times_M P^-)$. Using idempotents, we relate $\overline{M}$ to the Vinberg semigroup of G. The monoid $\overline{M}$ is used implicitly in the study of stratifications of Drinfeld's compactifications of the moduli stacks Bun_P and Bun_G.

1. Introduction

1.1. Motivation.

1.1.1. Let G be a connected reductive group over a perfect field k. Let $U(P)$ denote the unipotent radical of a parabolic subgroup P of G. Grosshans proved in [10] that the homogeneous space $G/U(P)$ is a quasi-affine variety and the algebra of regular functions $k[G/U(P)]$ is finitely generated.

In [1], Arzhantsev and Timashev consider affine embeddings of $G/U(P)$ and give a detailed description of the *canonical embedding* $G/U(P) \hookrightarrow \operatorname{Spec} k[G/U(P)]$ under the assumption that the characteristic of k is 0. They establish a bijection between these affine embeddings and certain normal algebraic monoids with group of units equal to the Levi factor $M = P/U(P)$. In particular, the canonical embedding corresponds to the monoid $\overline{M}$ defined as the closure of M in $\operatorname{Spec} k[G/U(P)]$. This construction, which we first learned from [2], defines an affine algebraic monoid $\overline{M}$ in any characteristic. It is not *a priori* clear, however, whether the monoid $\overline{M}$ is normal in positive characteristic.

One of the goals of this paper is to show that $\overline{M}$ is a normal algebraic monoid with group of units M in any characteristic, and to describe the combinatorial data it corresponds to under the classification of normal reductive monoids in [13, Theorem 5.4].

1.1.2. Let $\overline{G/U(P)}$ denote the spectrum of $k[G/U(P)]$. Then $\overline{G/U(P)}$ is an affine variety of finite type, and it plays a prominent role in the definition of Drinfeld's compactification $\widetilde{\mathrm{Bun}}_P$ of the moduli stack of P-bundles over a smooth complete curve. Drinfeld's compactification is used to define the geometric Eisenstein series functors in [6]. As Baranovsky observes in [2, §6], the monoid $\overline{M}$ is used implicitly when studying the stratification of $\widetilde{\mathrm{Bun}}_P$. More specifically, the closed subscheme $\mathrm{Gr}^+_M \subset \mathrm{Gr}_M$ of the affine Grassmannian (cf. [6, §6.2], [5, §1.6]) is just $(\overline{M}(O) \cap M(K))/M(O)$ inside $M(K)/M(O)$, where O is a complete discrete valuation ring with field of fractions K. The relative version of Gr^+_M becomes $\mathcal{H}^+_M$, the positive part of the Hecke stack (cf. [5, §1.8]).

The stack $\mathcal{H}_M^+$ is therefore the global model for the formal arc space of the embedding $M \hookrightarrow \overline{M}$, as considered in [4, §2]. We hope that studying the properties of $\overline{M}$ will provide a better understanding of $\mathcal{H}_M^+$.

1.1.3. If P^- is a parabolic subgroup opposite to P, then $G/U(P)$ is closely related to the more symmetrically defined variety $X_P = (G/U(P) \times G/U(P^-))/(P \cap P^-)$, which is also quasi-affine. This variety X_P is called a *boundary degeneration* of G in [16] (when P is not a Borel subgroup, X_P is an *intermediate* degeneration), and it is a central object in the geometric proof of Bernstein's Second Adjointness Theorem in the theory of $\mathfrak{p}$-adic groups given in [3]. We note that this proof and the space X_P are closely related to the study of geometric constant term (and Eisenstein series) functors in [7].

The boundary degeneration X_P and its affine closure $\overline{X}_P := \operatorname{Spec} k[X_P]$ may be recovered from the *Vinberg semigroup* corresponding to G. The Vinberg semigroup $\overline{G_{\mathrm{enh}}}$ is used to define the Drinfeld-Lafforgue compactification $\overline{\mathrm{Bun}}_G$ (resp. the Drinfeld-Lafforgue-Vinberg compactification VinBun_G) of the moduli stack Bun_G in [9, 17]. As one might expect, the positive part $\mathcal{H}_M^+$ of the Hecke stack appears in the stratification of $\overline{\mathrm{Bun}}_G$ (resp. VinBun_G), where P ranges over all conjugacy classes of parabolic subgroups, assuming that G is split. In this article we attempt to explain the relations between $\overline{M}$, $\overline{G/U(P)}$, $\overline{X}_P$, and $\overline{G_{\mathrm{enh}}}$ in hopes that it will elucidate the geometry underlying the aforementioned stratifications.

1.1.4. In [15], Sakellaridis fixes a strictly convex cone in the $\mathbb{Q}$-vector space spanned by the coweights of a split maximal torus T in G in order to "expand power series" on the boundary degeneration X_P, under the assumption that the characteristic of k is 0. This cone is precisely the dual of what we call the *Renner cone* of $\overline{M}$. Thus the combinatorial description of $\overline{M}$ provides a first step towards generalizing the results of [15] to arbitrary characteristic.

The description of $\overline{M}$ is also of interest in the study of those local unramified automorphic L-functions associated to certain "basic functions" on $\overline{M}$ in the spirit of [4]. Such functions are considered in [19] in relation to the asymptotics map[1] and inversion of intertwining operators. The study of $\overline{M}$, and more generally of the intermediate boundary degenerations X_P, is needed in [19] to generalize the results of [8], which treats the case when $G = \mathrm{SL}(2)$.

1.2. **Contents.** In §2, we recall the classification of normal reductive monoids proved by L. Renner. Given a reductive group and certain combinatorial data (what we call a *Renner cone*), we construct the associated normal algebraic monoid.

In §3, we define the normal reductive monoid $\overline{M}$ associated to a parabolic subgroup P of G. The group of units of $\overline{M}$ is the Levi factor M of P. We first give a combinatorial definition of $\overline{M}$ following Renner's classification. We then show in §3.2 that this monoid may be realized as a retract of $\overline{G/U(P)}$, the spectrum of regular functions on the quasi-affine variety $G/U(P)$. Lastly in §3.3 we describe $\overline{M}$ using the Tannakian formalism. This Tannakian description shows how $\overline{M}$ is used implicitly in [6], [5].

In §4, we first recall the definition of the boundary degeneration X_P associated to a pair of opposite parabolics. We show that $\overline{G/U(P)}$ is a retract (and hence a closed subscheme) of $\overline{X}_P := \operatorname{Spec} k[X_P]$. Using the relation between the boundary degeneration and the Vinberg semigroup of G (i.e., the enveloping semigroup of G), we give another definition of the reductive monoid $\overline{M}$ using the existence of a certain idempotent in the Vinberg semigroup.

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[1] The asymptotics map, defined in [15, 16], coincides with the dual of the Bernstein map defined in [3].

1.3. **Conventions.** Let k be a perfect field of arbitrary characteristic. All schemes considered will be k-schemes. For a scheme S, let $k[S]$ denote the ring of regular functions $\Gamma(S, \mathcal{O}_S)$.

Fix an algebraic closure $\bar{k}$ of k, and let $\mathrm{Gal}(\bar{k}/k)$ denote its Galois group. For a k-scheme S, let $S_{\bar{k}}$ denote the base change $S \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$, and let $\bar{k}[S] := \Gamma(S_{\bar{k}}, \mathcal{O}_{S_{\bar{k}}})$.

1.3.1. *The group G.* Let G be a connected reductive group over k. Let T denote its *abstract* Cartan and W the corresponding Weyl group. We will denote by $\check{\Lambda}$ (resp. Λ) the weight (resp. coweight) lattice of $T_{\bar{k}}$, which is a $\mathrm{Gal}(\bar{k}/k)$-module.

The semigroup of dominant coweights (resp., weights) will be denoted by Λ_G^+ (resp., by $\check{\Lambda}_G^+$). The set of vertices of the Dynkin diagram of G will be denoted by Γ_G; for each $i \in \Gamma_G$ there corresponds a simple coroot α_i and a simple root $\check{\alpha}_i$. We denote the non-negative integral span of the set of positive coroots (resp. roots) by Λ_G^{pos} (resp. $\check{\Lambda}_G^{\mathrm{pos}}$). For $\lambda, \mu \in \Lambda$ we will write that $\lambda \geq \mu$ if $\lambda - \mu \in \Lambda_G^{\mathrm{pos}}$, and similarly for $\check{\Lambda}_G^{\mathrm{pos}}$. Let w_0 denote the longest element in the Weyl group of G.

Let P be a parabolic subgroup of G. Let $U(P)$ be its unipotent radical and $M := P/U(P)$ the Levi factor. We use P to identify the abstract Cartan of M with T and let $W_M \subset W$ denote the corresponding Weyl group. There is a subdiagram $\Gamma_M \subset \Gamma_G$. We will denote by $\Lambda_M^{\mathrm{pos}} \subset \Lambda_G^{\mathrm{pos}}$, $\Lambda_M^+ \supset \Lambda_G^+$, $\geq_M$, $w_0^M \in W_M$, etc. the corresponding objects for M.

1.3.2. Let $\mathrm{Rep}(G)$ denote the abelian category of finite-dimensional G-modules. This category admits a forgetful functor to the abelian category of k-vector spaces. We define the functor

$$\mathrm{ind}_P^G : \mathrm{Rep}(P) \to \mathrm{Rep}(G)$$

as in [11, §I.3.3]. For a P-module $\overline{V}$, the induced module $\mathrm{ind}_P^G(\overline{V}) = (k[G] \otimes_k \overline{V})^P$ is finite-dimensional by properness of G/P. The functor ind_P^G is right adjoint to the restriction functor (cf. [11, Proposition I.3.4]). We also denote by ind_P^G the corresponding functor $\mathrm{Rep}(M) \to \mathrm{Rep}(G)$, where an M-module is considered as a P-module with trivial $U(P)$-action.

To a dominant weight $\check{\lambda} \in \check{\Lambda}_G^+$ one attaches the Weyl $G_{\bar{k}}$-module $\Delta(\check{\lambda})$, the dual Weyl module $\nabla(\check{\lambda})$, and the irreducible $G_{\bar{k}}$-module $L(\check{\lambda})$ of highest weight $\check{\lambda}$.

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2. Recollections on normal reductive monoids

In this section we give a brief review of the classification of normal reductive monoids (i.e., normal, irreducible, affine algebraic monoids whose group of units is reductive), which is proved in [13, Theorem 5.4] by L. Renner. In [13], the base field is assumed to be algebraically closed, but the statements easily generalize to the case of a perfect base field by Galois descent.

To keep notation consistent with the rest of the article, we consider a connected reductive group M over k. Let T denote its *abstract* Cartan and W_M the corresponding Weyl group.

2.1. **Renner cones.** We denote by $\check{\Lambda}$ the weight lattice of $T_{\bar{k}}$ (i.e., the lattice of characters). Let $\check{\Lambda}^{\mathbb{Q}} := \check{\Lambda} \otimes \mathbb{Q}$, which is a $\mathbb{Q}$-vector space with a $\mathrm{Gal}(\bar{k}/k)$-action.

A *Renner cone* is a convex rational polyhedral cone in $\check{\Lambda}^{\mathbb{Q}}$ that is stable under the actions of W_M and $\mathrm{Gal}(\bar{k}/k)$. As the name suggests, the theorem of L. Renner shows that normal algebraic monoids with group of invertible elements M bijectively correspond to Renner cones generating $\check{\Lambda}^{\mathbb{Q}}$ as a vector space. The correspondence is as follows:

Let $\overline{M}$ be a reductive monoid with group of units M. Fix a Borel subgroup $B \subset M_{\bar{k}}$ and a Cartan subgroup (i.e., maximal torus) $T_{\mathrm{sub},\bar{k}} \subset B$, both defined over $\bar{k}$. This gives an identification of $T_{\mathrm{sub},\bar{k}}$ with the abstract Cartan $T_{\bar{k}}$. Consider the cone $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$ corresponding by [12] to the closure of $T_{\mathrm{sub},\bar{k}}$ in $\overline{M}_{\bar{k}}$. The pairs $(T_{\mathrm{sub},\bar{k}}, B)$ of a Cartan subgroup contained in a Borel subgroup are all conjugate by $M(\bar{k})$. Since M acts on $\overline{M}$ by conjugation, $\check{C}$ does not depend on the choice of $(T_{\mathrm{sub},\bar{k}}, B)$. The Weyl group of M acts on $T_{\mathrm{sub},\bar{k}}$ through the normalizer of $T_{\mathrm{sub},\bar{k}}$ in M, so $\check{C}$ is preserved by the action of W_M on $\check{\Lambda}^{\mathbb{Q}}$. The action of $\mathrm{Gal}(\bar{k}/k)$ on $G_{\bar{k}}$ induces an action on the set of pairs $(T_{\mathrm{sub},\bar{k}}, B)$. Since $\check{C}$ is canonically defined independently of the choice of $(T_{\mathrm{sub},\bar{k}}, B)$, the Galois action preserves $\check{C}$. Therefore $\check{C}$ is a Renner cone, and it is the Renner cone corresponding to $\overline{M}$.

2.1.1. Let $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$ be a Renner cone. We will construct the corresponding normal reductive monoid $\overline{M}$. Let us choose a Cartan subgroup (i.e., maximal torus) T_{sub} of M, defined over k. The construction of $\overline{M}$ will not depend on this choice.

2.2. **The monoid $\overline{T_{\mathrm{sub}}}$.** The characters $\check{\Lambda}$ form a basis of $\bar{k}[T]$. Let R' denote the subalgebra of $\bar{k}[T]$ spanned by the characters in $\check{C} \cap \check{\Lambda}$. The choice of a Borel $B \subset M_{\bar{k}}$ containing $T_{\mathrm{sub},\bar{k}}$ gives an isomorphism $T_{\mathrm{sub},\bar{k}} \cong T_{\bar{k}}$. All such Borel subgroups are conjugate by the normalizer of $T_{\mathrm{sub},\bar{k}}$ in $M_{\bar{k}}$. The subalgebra R' is preserved by the action of the Weyl group on T, so it defines a corresponding subalgebra $R \subset \bar{k}[T_{\mathrm{sub}}]$, which does not depend on the choice of a Borel subgroup.

2.2.1. Since $\check{C}$ is Galois stable, so is the subalgebra R. Set $\overline{T_{\mathrm{sub}}} := \mathrm{Spec}(R^{\mathrm{Gal}(\bar{k}/k)})$.

Lemma 2.2.2. *(i) $\overline{T_{\mathrm{sub}}}$ is a normal algebraic variety containing T_{sub} as a dense open subvariety.*

(ii) $\overline{T_{\mathrm{sub}}}$ has a (unique) monoidal structure extending the group structure on T_{sub}.

Proof. By Galois descent, it suffices to check the statements over $\bar{k}$, and we have $\bar{k}[\overline{T_{\mathrm{sub}}}] = R$. The submonoid $\check{C} \cap \check{\Lambda}$ is finitely generated and generates $\check{\Lambda}$ as a group. Moreover the submonoid is *saturated* (i.e., it is the intersection of a rational cone with the lattice). Statement (i) follows from [12, Ch. 1, Thm. 1].

To prove statement (ii), one must show that the coproduct map $k[T_{\mathrm{sub}}] \to k[T_{\mathrm{sub}}] \otimes k[T_{\mathrm{sub}}]$ sends the subalgebra R to $R \otimes R$. This is clear because $R \otimes \bar{k}$ has a basis consisting of characters of $T_{\mathrm{sub},\bar{k}}$. □

2.3. **The monoid $\overline{M}$.** We will define a normal algebraic monoid $\overline{M}$ with group of units M such that the closure of T_{sub} in $\overline{M}$ equals $\overline{T_{\mathrm{sub}}}$. The monoid $\overline{M}$ will be the spectrum of a certain subalgebra A of the algebra of regular functions on M.

2.3.1. *The algebra A.* Let A denote the algebra of all $f \in k[M]$ such that for any $m_1, m_2 \in M(\bar{k})$ the function

$$t \mapsto f(m_1 t m_2)$$

belongs to the algebra R defined in §2.2. Since all Cartan subgroups of $M_{\bar{k}}$ are $M(\bar{k})$-conjugate, A does not depend on the choice of the subgroup $T_{\mathrm{sub}} \subset M$.

Proposition 2.3.2. *(i) A is a sub-bialgebra of the Hopf algebra $k[M]$.*

(ii) The map $M \to \mathrm{Spec}\, A$ is an open embedding.

(iii) A is an integrally closed domain.

(iv) The algebra A is finitely generated.

(v) The homomorphism $A \to k[\overline{T_{\mathrm{sub}}}]$ that takes a function to its restriction to T_{sub} is surjective.

Proof. All statements can be checked after base change to $\bar{k}$, so we will assume that k is algebraically closed.

Let A' denote the subalgebra of $k[M]$ generated by the matrix coefficients of a *finite* collection of Weyl[2] M-modules whose highest weights belong to $\check{C} \cap \check{\Lambda}^+_G$ and generate $\check{\Lambda}^+_G$ as a semigroup. The following properties of A' are easy to check:

(a) $A' \subset A$;

(a′) if k has characteristic 0 then $A' = A$;

(a″) the morphism $M \to \operatorname{Spec} A'$ is an open embedding;

(b) the composition $A' \hookrightarrow A \to R$ is surjective;

(c) the algebra A' is finitely generated.

The proof of statement (i) in the proposition is standard. Since $A' \subset A \subset k[M]$, property (a″) implies statement (ii). Statement (iii) follows from the normality of M and the normality part of Lemma 2.2.2(i). Statement (v) follows from (b).

Let A'' denote the integral closure of A' in the function field of M. Without any assumptions on the characteristic of k, we claim that $A = A''$. By (ii)-(iii), it suffices to check that A is contained in the localization O of A'' at any codimension 1 prime. Let K denote the field of fractions of O, which is also the field of rational functions on M. Then the normalization map $\operatorname{Spec} A'' \to \operatorname{Spec} A'$ induces a map $f' : \operatorname{Spec} O \to \operatorname{Spec} A'$ with $f'(\operatorname{Spec} K) \subset M$. We wish to lift f' to a morphism $f : \operatorname{Spec} O \to \operatorname{Spec} A$. Since

$$M(K) = M(O) \cdot T_{\mathrm{sub}}(K) \cdot M(O)$$

we can assume that $f'(\operatorname{Spec} K) \subset T_{\mathrm{sub}}(K)$. Then the existence of f follows from (b), which says that the closure of T_{sub} in $\operatorname{Spec} A$ maps isomorphically onto the closure of T_{sub} in $\operatorname{Spec} A'$. Therefore $A = A''$, and statement (iv) now follows. □

2.3.3. *The algebraic monoid $\overline{M}$.* Now set $\overline{M} := \operatorname{Spec} A$.

By Proposition 2.3.2, $\overline{M}$ is a normal affine algebraic monoid equipped with an open embedding $M \hookrightarrow \overline{M}$ with dense image. By part (v) of the proposition, the closed embedding $T_{\mathrm{sub}} \hookrightarrow M$ extends to a closed embedding $\overline{T_{\mathrm{sub}}} \hookrightarrow \overline{M}$. By construction, the Renner cone corresponding to $\overline{M}$ is $\check{C}$.

Since $\overline{M}$ is an irreducible monoid and M is an open dense subgroup, M is necessarily the group of units of $\overline{M}$. The classification theorem of L. Renner ([13, Theorem 5.4]) says that every normal algebraic monoid with group of units M is isomorphic to a monoid $\overline{M}$ of the above form.

3. The monoid associated to a parabolic subgroup

Let P be a parabolic subgroup of G with Levi quotient $M := P/U(P)$. We will define a canonical normal reductive monoid $\overline{M}$ with group of units M. This monoid appears implicitly in [6, 5], and it is explicitly considered in [1, §3.3] (in characteristic 0) and in [2, §6].

We identify the abstract Cartans of G and M as follows: for a Borel subgroup $B_M \subset M_{\bar{k}}$, the subgroup $B := B_M U(P) \subset G_{\bar{k}}$ is a Borel subgroup, and $T_{\bar{k}} = B/U(B) = B_M/U(B_M)$.

3.1. The Renner cone of $\overline{M}$.

We first give a combinatorial definition of $\overline{M}$ using Renner's classification, recalled in §2. We will specify the Renner cone $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$, from which one constructs $\overline{M}$ as in §2.

[2]One can also take dual Weyl modules.

3.1.1. *The submonoid* $\Lambda^{\mathrm{pos}}_{U(P)}$. Let $\Lambda^{\mathrm{pos}}_{U(P)} \subset \Lambda$ denote the non-negative integral span of the positive coroots of G that are not coroots of M. The submonoid $\Lambda^{\mathrm{pos}}_{U(P)}$ is stable under the actions of W_M and $\mathrm{Gal}(\bar{k}/k)$ because M is defined over k. We use representation theory to show certain properties of $\Lambda^{\mathrm{pos}}_{U(P)}$ below.

Let $\check{G}$ (resp. $\check{M}$) denote the Langlands dual group of G (resp. M) over $\mathbb{C}$. Fix a maximal torus and a Borel subgroup containing it in the split group $\check{G}$. Then we may consider $\check{M}$ as a Levi subgroup of $\check{G}$. Let $\check{\mathfrak{u}}_P$ denote the nilpotent Lie algebra corresponding to the positive coroots of G that are not coroots of M. Then the symmetric algebra $\mathrm{Sym}(\check{\mathfrak{u}}_P)$ is a locally finite $\check{M}$-module by the adjoint action, and its set of weights equals $\Lambda^{\mathrm{pos}}_{U(P)}$.

Lemma 3.1.2. *Let $\lambda, \lambda' \in \Lambda^+_M$ with $\lambda \le_M \lambda'$. If $\lambda' \in \Lambda^{\mathrm{pos}}_{U(P)}$, then $\lambda \in \Lambda^{\mathrm{pos}}_{U(P)}$.*

Proof. We have a decomposition of $\mathrm{Sym}(\check{\mathfrak{u}}_P)$ into irreducible highest weight $\check{M}$-modules $L_{\check{M}}(\gamma)$. Therefore λ' is a weight in $L_{\check{M}}(\gamma)$ for some $\gamma \in \Lambda^+_M$, and all the weights of $L_{\check{M}}(\gamma)$ lie in $\Lambda^{\mathrm{pos}}_{U(P)}$. Since $\lambda \in \Lambda^+_M$ and $\lambda \le_M \lambda' \le_M \gamma$, we deduce that λ is also a weight of $L_{\check{M}}(\gamma)$. Therefore $\lambda \in \Lambda^{\mathrm{pos}}_{U(P)}$. □

Lemma 3.1.3. *The subset $\Lambda^{\mathrm{pos}}_{U(P)} \subset \Lambda$ is equal to the intersection of $w(\Lambda^{\mathrm{pos}}_G)$ for all $w \in W_M$. Consequently, $\Lambda^{\mathrm{pos}}_{U(P)} \cap (-\Lambda^+_M) = \Lambda^{\mathrm{pos}}_G \cap (-\Lambda^+_M)$.*

Proof. Observe that $\Lambda^{\mathrm{pos}}_{U(P)}$ is W_M-stable and hence contained in $w(\Lambda^{\mathrm{pos}}_G)$ for all $w \in W_M$. To prove containment in the other direction, let $\lambda \in \bigcap_{w \in W_M} w(\Lambda^{\mathrm{pos}}_G)$. Replacing λ by an element in the same W_M-orbit, we may assume that $\lambda \in -\Lambda^+_M$. By assumption $\lambda \in \Lambda^{\mathrm{pos}}_G$, so we can write $\lambda = \lambda_1 + \lambda_2$ where λ_1 is a linear combination of α_i for $i \in \Gamma_M$ and $\lambda_2 \in \Lambda^{\mathrm{pos}}_{U(P)}$ is a linear combination of α_j for $j \in \Gamma_G \setminus \Gamma_M$. Note that $\lambda_2 \in \Lambda^{\mathrm{pos}}_{U(P)} \cap (-\Lambda^+_M)$ and $\lambda \ge_M \lambda_2$. Then $w_0^M \lambda_2 \in \Lambda^{\mathrm{pos}}_{U(P)} \cap \Lambda^+_M$ and $w_0^M \lambda \le_M w_0^M \lambda_2$. Lemma 3.1.2 implies that $w_0^M \lambda \in \Lambda^{\mathrm{pos}}_{U(P)}$, and hence $\lambda \in \Lambda^{\mathrm{pos}}_{U(P)}$. One deduces the second statement of the lemma from the first because $\lambda \in -\Lambda^+_M$ satisfies $\lambda \le_M w\lambda$ for all $w \in W_M$. □

Lemma 3.1.4. *The submonoid $W_M \cdot \check{\Lambda}^+_G \subset \check{\Lambda}$ is dual to $\Lambda^{\mathrm{pos}}_{U(P)}$, i.e.,*

$$W_M \cdot \check{\Lambda}^+_G = \{\check{\lambda} \in \check{\Lambda} \mid \langle \check{\lambda}, \mu \rangle \ge 0 \text{ for all } \mu \in \Lambda^{\mathrm{pos}}_{U(P)}\}. \tag{3.1}$$

Proof. Let $(\Lambda^{\mathrm{pos}}_{U(P)})^\vee$ equal the r.h.s. of (3.1), which is evidently W_M-stable. If we consider an element in $(\Lambda^{\mathrm{pos}}_{U(P)})^\vee \cap \check{\Lambda}^+_M$, then it pairs with positive coroots of M to non-negative integers since the element is $\check{M}$-dominant, and it pairs with all other positive coroots of G to non-negative integers by definition of the dual. Thus $(\Lambda^{\mathrm{pos}}_{U(P)})^\vee \cap \check{\Lambda}^+_M = \check{\Lambda}^+_G$, which implies that $(\Lambda^{\mathrm{pos}}_{U(P)})^\vee$ is the union of $w(\check{\Lambda}^+_G)$ for all $w \in W_M$. □

Corollary 3.1.5. *The submonoid $W_M \cdot \check{\Lambda}^+_G$ is saturated in $\check{\Lambda}$.*

3.1.6. *The Renner cone* $\check{C}$. Set $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$ to be the convex rational polyhedral cone generated by $W_M \cdot \check{\Lambda}^+_G$. Lemma 3.1.4 implies that $\check{C}$ is preserved by the actions of W_M and $\mathrm{Gal}(\bar{k}/k)$, and Corollary 3.1.5 says that $\check{C} \cap \check{\Lambda} = W_M \cdot \check{\Lambda}^+_G$.

3.1.7. *Definition of* $\overline{M}$. Set $\overline{M}$ to be the normal reductive monoid with Renner cone $\check{C}$ constructed in Proposition 2.3.2. We will use this notation for the rest of the article.

3.2. **Relation to $\overline{G/U(P)}$.** In this subsection, we show (see Corollary 3.2.9) that $\overline{M}$ is isomorphic to the monoid constructed in [1, §3.3] and [2, §6].

First we recall some facts about the homogeneous space $G/U(P)$.

3.2.1. A scheme S is *strongly quasi-affine* if the canonical morphism $S \to \operatorname{Spec} k[S]$ is an open embedding and $k[S]$ is a finitely generated k-algebra.

F. D. Grosshans proved that the quotient variety $G/U(P)$ is strongly quasi-affine in [10]. Let $\overline{G/U(P)} = \operatorname{Spec} k[G/U(P)]$, where $k[G/U(P)]$ is the subalgebra of right $U(P)$-invariant regular functions on G.

3.2.2. *Weights of* $k[G/U(P)]$. The Levi factor $M := P/U(P)$ acts on $G/U(P)$ from the right. Therefore we can consider $k[G/U(P)]$ as an M-module and ask what is the set of weights[3] of this module with respect to the abstract Cartan of M.

Lemma 3.2.3. *The set of weights of the M-module $k[G/U(P)]$ equals $W_M \cdot \check{\Lambda}_G^+ \subset \check{\Lambda}$.*

Proof. We may assume that k is algebraically closed. Choose a Borel subgroup B contained in P (so $B/U(P)$ is a Borel subgroup of M) and let $T_{\text{sub}} \subset B$ be a maximal torus, which we identify with its image in M. The weights of $k[G/U(P)]$ are the T_{sub}-eigenvalues with respect to right translations. Let $k[G/U(P)]_{\check{\gamma}}$, $\check{\gamma} \in \check{\Lambda}$, denote a weight space.

Note that $k[G/U(P)]_{\check{\gamma}}$ is a G-module by left translation. Let B^- denote the opposite Borel subgroup so that $B \cap B^- = T_{\text{sub}}$. By unipotence of $U(B^-)$, we deduce that $\check{\gamma}$ is a weight of $k[G/U(P)]$ if and only if $k[G/U(P)]_{\check{\gamma}}^{U(B^-)} \neq 0$. Hence we are reduced to studying the weight spaces of $k[G/U(P)]^{U(B^-)}$. By considering the T-action by left translation, we have decompositions

$$k[U(B^-)\backslash G] = \bigoplus_{\check{\lambda} \in \check{\Lambda}_G^+} \nabla(\check{\lambda}), \qquad k[U(B^-)\backslash G]^{U(P)} = \bigoplus_{\check{\lambda} \in \check{\Lambda}_G^+} \nabla(\check{\lambda})^{U(P)}$$

where $U(P)$ acts by right translation. Since $U(B^-)P$ is dense in G, the restriction from G to P gives an injection

$$\nabla(\check{\lambda})^{U(P)} \hookrightarrow \nabla_M(\check{\lambda}),$$

where $\nabla_M(\check{\lambda})$ is the dual Weyl M-module.

We now prove the 'only if' direction of the lemma. Suppose that $\check{\gamma}$ is a weight of $k[G/U(P)]$. Then $\check{\gamma}$ must be a weight of $\nabla_M(\check{\lambda})$ for some $\check{\lambda} \in \check{\Lambda}_G^+$. There exists $w \in W_M$ such that $w(\check{\gamma}) \in \check{\Lambda}_M^+$. Since the set of weights of $\nabla_M(\check{\lambda})$ is W_M-stable, $w(\check{\gamma})$ is also a weight. Hence $w(\check{\gamma}) \leq_M \check{\lambda}$. Since $\langle \check{\alpha}_i, \alpha_j \rangle \leq 0$ for $i \in \Gamma_M, j \in \Gamma_G \setminus \Gamma_M$, we deduce that $w(\check{\gamma}) \in \check{\Lambda}_G^+$. This proves the 'only if' direction of the lemma.

Conversely, suppose $\check{\gamma}$ is a weight such that $w(\check{\gamma}) \in \check{\Lambda}_G^+$ for some $w \in W_M$. Then $\check{\lambda} := w(\check{\gamma})$ is the highest weight in $\nabla(\check{\lambda})^{U(P)}$. Since the set of weights of an M-module is W_M-stable, we conclude that $\check{\gamma}$ is a weight of $k[G/U(P)]$. ∎

Corollary 3.2.4. *For any G-module V, the weights of the M-module $V^{U(P)}$ are a subset of $W_M \cdot \check{\Lambda}_G^+$.*

Proof. Any finite dimensional G-module V is a submodule of a direct sum of regular representations $k[G]$, so the weights of $V^{U(P)}$ are a subset of the weights of $k[G/U(P)]$. ∎

[3] Let V be an M-module over k. Choose a Borel subgroup $B_M \subset M_{\bar{k}}$ and a Cartan subgroup $T_{\text{sub},\bar{k}} \subset B_M$, which is isomorphic to $T_{\bar{k}} = B_M/U(B_M)$. We say that the set of weights of V is the set of $T_{\text{sub},\bar{k}}$-eigenvalues of $V \otimes \bar{k}$. This set does not depend on the choice of $(T_{\text{sub},\bar{k}}, B_M)$, so it can be considered as a subset of $\check{\Lambda}$, which is preserved by W_M and $\operatorname{Gal}(\bar{k}/k)$.

3.2.5. *The closure of M in $\overline{G/U(P)}$.* The subgroup $P \subset G$ induces a closed embedding

$$M = P/U(P) \hookrightarrow G/U(P), \tag{3.2}$$

i.e., we embed M in $G/U(P)$ by the right M-action on $1 \in G$. Then the closure of M in $\overline{G/U(P)}$ has the structure of an irreducible algebraic monoid[4], and the right action of M on $G/U(P)$ extends to an action of this monoid on $\overline{G/U(P)}$. We claim that the normalization of this monoid is isomorphic to the monoid $\overline{M}$ defined in §3.1.7:

Lemma 3.2.6. *The embedding* (3.2) *extends to a finite map* $\overline{M} \to \overline{G/U(P)}$.

Proof. Let T_{sub} be a Cartan subgroup of M and embed $T_{\text{sub}} \hookrightarrow G/U(P)$ using (3.2). Let $\overline{T_{\text{sub}}}$ denote the closure of T_{sub} in $\overline{G/U(P)}$. By the classification of normal reductive monoids in [13, Theorem 5.4], it suffices to show that the cone corresponding by [12] to $\overline{T_{\text{sub}}}$ is the Renner cone $\check{C}$ of $\overline{M}$.

By definition, $\overline{T_{\text{sub}}}$ is the spectrum of the image of the restriction map $k[G/U(P)] \to k[T_{\text{sub}}]$. This map is equivariant with respect to right translations by T_{sub}, so $\bar{k}[\overline{T_{\text{sub}}}]$ decomposes into weight spaces. Let $\check{\gamma}$ be a weight of $\bar{k}[G/U(P)]$. By left translation by G, one can find $f \in \bar{k}[G/U(P)]_{\check{\gamma}}$ such that $f(1) = 1$. Therefore the weights of $k[\overline{T_{\text{sub}}}]$ coincide with the weights of $k[G/U(P)]$, and the claim follows from Lemma 3.2.3. □

3.2.7. Fix a parabolic subgroup $P^- \subset G$ opposite to P. For the rest of this section we will identify M with the Levi subgroup $P \cap P^-$.

Theorem 3.2.8. *The composition*

$$\overline{M} \to \overline{G/U(P)} \to \operatorname{Spec} k[G]^{U(P^-) \times U(P)}$$

is an isomorphism, where $U(P^-) \times U(P)$ *acts on* $k[G]$ *by left and right translations, respectively.*

Note that $\operatorname{Spec} k[G]^{U(P^-) \times U(P)}$ is the affine GIT quotient of $\overline{G/U(P)}$ by the left action of $U(P^-) \subset G$.

Corollary 3.2.9. *The (unique) map* $\overline{M} \to \overline{G/U(P)}$ *extending the embedding* (3.2) *is a retract. In particular, it is a closed embedding.*

Proof. The fact that $\overline{M}$ is a retract of $\overline{G/U(P)}$ follows immediately from the isomorphism in Theorem 3.2.8. To prove that it is a closed subscheme, it suffices to show that the algebra map $k[G/U(P)] \to k[\overline{M}]$ is surjective. The theorem implies that the subalgebra $k[G]^{U(P^-) \times U(P)} \subset k[G/U(P)]$ surjects onto $k[\overline{M}]$. □

For the purpose of proving Theorem 3.2.8, let $\tilde{M} = \operatorname{Spec} k[G]^{U(P^-) \times U(P)}$. The actions of M on G by left and right translations induce M-actions on $\tilde{M}$. We have a canonical $M \times M$ equivariant map $G \to \tilde{M}$.

Lemma 3.2.10. *The composition* $M \to G \to \tilde{M}$ *is an open embedding.*

Proof. We may check the assertion after base change to $\bar{k}$, so we assume k is algebraically closed. Choose Borel subgroups $B \subset P$ and $B^- \subset P^-$ such that $T_{\text{sub}} := B \cap B^- \subset M$ is a maximal torus. Let

$$\tilde{T} = \operatorname{Spec} k[G]^{U(B^-) \times U(B)}.$$

Since $k[G]^{U(B^-)}$ has a decomposition into dual Weyl G-modules, one deduces that $k[\tilde{T}]$ has a basis formed by $f_{\check{\lambda}}$ for $\check{\lambda} \in \check{\Lambda}^+_G$, where $f_{\check{\lambda}}(t) = \check{\lambda}(t)$, $t \in T_{\text{sub}}$. From this explicit description, one sees that $\tilde{T}$ is a toric variety containing T_{sub} as a dense open subscheme.

[4]This monoid is denoted by M_+ in [2].

Consider the composition $G \to \tilde{M} \to \tilde{T}$ and let $\overset{\circ}{G} \subset G$ denote the preimage of $T_{\text{sub}} \subset \tilde{T}$. Then the preimage of T_{sub} in $\tilde{M}$, which we denote $\overset{\circ}{\tilde{M}}$, is equal to $\operatorname{Spec} k[\overset{\circ}{G}]^{U(P^-) \times U(P)}$. Observe that $B^-B = U(B^-) \times T_{\text{sub}} \times U(B)$ is an open affine subset contained in $\overset{\circ}{G}$. Let us show that $\overset{\circ}{G} = B^-B$. By definition, $\overset{\circ}{G}$ consists of $g \in G$ such that $f_{\check{\lambda}}(g) \neq 0$ for all dominant weights $\check{\lambda}$. By the Bruhat decomposition, it suffices to show that if w belongs to the normalizer of T_{sub} but not to T_{sub} (i.e., w corresponds to a nontrivial element of W), then there exists $\check{\lambda}$ with $f_{\check{\lambda}}(w) = 0$. Indeed, for a dominant regular weight $\check{\lambda}$ we have $w\check{\lambda} \neq \check{\lambda}$. Thus the left and right T-actions on $w^{-1}f_{\check{\lambda}}$ do not have the same weight, which implies that $f_{\check{\lambda}}(w) = (w^{-1}f_{\check{\lambda}})(1) = 0$.

Let $B_M = B/U(P) = B \cap M$ and $B_M^- = B^-/U(P^-) = B^- \cap M$. From the equality $\overset{\circ}{G} = U(B^-) \times T_{\text{sub}} \times U(B)$ we deduce that $\overset{\circ}{\tilde{M}} = U(B_M^-) \times T_{\text{sub}} \times U(B_M)$ is an open dense subset of both M and $\tilde{M}$. Using left (or right) translations by M, we deduce that the whole group M is an open subset of $\tilde{M}$. □

The fraction field of $\tilde{M}$ is contained in[5] the field of invariants $k(G)^{U(P^-) \times U(P)}$. Thus normality of G implies normality of $\tilde{M}$. Therefore Lemma 3.2.10 implies that $\tilde{M}$ is a normal reductive monoid with group of units M.

Proof of Theorem 3.2.8. Let T_{sub} be a Cartan subgroup of $M \subset G$. Since $\tilde{M}$ is a normal reductive monoid with group of units M, it is determined by the closure of T_{sub} in $\tilde{M}$, which is the spectrum of the algebra

$$\tilde{R} := \operatorname{Im}(k[G]^{U(P^-) \times U(P)} \to k[T_{\text{sub}}]).$$

By unipotence of $U(P^-)$, the algebra $\tilde{R}$ is the image of the restriction map $k[G/U(P)] \to k[T_{\text{sub}}]$. Therefore $\operatorname{Spec} \tilde{R}$ is the closure of T_{sub} in $\overline{G/U(P)}$. By the proof of Lemma 3.2.6, this is also the closure of T_{sub} in $\overline{M}$. Since $\tilde{M}$ and $\overline{M}$ are both normal algebraic monoids with unit group M, the classification of normal reductive monoids ([13, Theorem 5.4]) implies that the map $\overline{M} \to \tilde{M}$ is an isomorphism. □

3.3. **Tannakian description of $\overline{M}$.** Let $\operatorname{Rep}(M)$ denote the monoidal category of finite-dimensional representations of M. Similarly, one has the monoidal category $\operatorname{Rep}(\overline{M})$. Since M is schematically dense in $\overline{M}$, the monoidal functor

$$\operatorname{Rep}(\overline{M}) \to \operatorname{Rep}(M)$$

corresponding to $M \hookrightarrow \overline{M}$ is fully faithful. So we can consider $\operatorname{Rep}(\overline{M})$ as a full subcategory of $\operatorname{Rep}(M)$.

3.3.1. The usual Tannakian formalism describes $\overline{M}$ in terms of $\operatorname{Rep}(\overline{M})$. Namely, for a test scheme S, an element of the monoid $\operatorname{Hom}(S, \overline{M})$ is a collection of assignments

$$\overline{V} \in \operatorname{Rep}(\overline{M}) \rightsquigarrow m_{\overline{V}} \in \operatorname{End}_{\mathcal{O}_S}(\overline{V} \otimes \mathcal{O}_S),$$

compatible with morphisms $\overline{V}_1 \to \overline{V}_2$ in $\operatorname{Rep}(\overline{M})$ and such that $m_{\overline{V}_1 \otimes \overline{V}_2} = m_{\overline{V}_1} \otimes m_{\overline{V}_2}$. The multiplication in $\operatorname{Hom}(S, \overline{M})$ corresponds to the multiplication in $\operatorname{End}_{\mathcal{O}_S}(\overline{V} \otimes \mathcal{O}_S)$.

Our goal is to prove Proposition 3.3.4 below, which describes the subcategory $\operatorname{Rep}(\overline{M})$.

[5]In fact one can show that the fraction field of $k[G]^{U(P^-) \times U(P)}$ is equal to $k(G)^{U(P)^- \times U(P)}$.

3.3.2. *Description of* $\operatorname{Rep}(\overline{M})$. Fix a parabolic subgroup $P^- \subset G$ opposite to P, and identify the Levi subgroup $P \cap P^-$ with M.

For an M-module $\overline{V}$, we consider an element $f \in \operatorname{ind}_{P^-}^G(\overline{V})$ as a regular map $G \to \overline{V}$ (cf. [11, §I.3.3]) satisfying $f(gm\bar{u}) = m^{-1}f(g)$ for all $\bar{k}$-points $g \in G, m \in M, \bar{u} \in U(P^-)$. Using this description, evaluation at 1 in G defines an M-morphism $\operatorname{ind}_{P^-}^G(\overline{V}) \to \overline{V}$.

Lemma 3.3.3. *Let* $\overline{V} \in \operatorname{Rep}(\overline{M})$. *Then evaluation at* 1 *induces an isomorphism*

$$\operatorname{ind}_{P^-}^G(\overline{V})^{U(P)} \to \overline{V}. \tag{3.3}$$

Proof. Since $U(P)P^-$ is a dense open subset of G, the map (3.3) is injective. Let $v \in \overline{V}$. Then we can define a morphism $f : U(P) \times M \times U(P^-) \cong U(P)P^- \to \overline{V}$ by $f(um\bar{u}) = m^{-1}v$ for $m \in M$, $u \in U(P)$, $\bar{u} \in U(P^-)$. For any $\xi \in \overline{V}^*$, the pairing $\langle \xi, f(um\bar{u})\rangle = \langle \xi, m^{-1}v\rangle$ extends to a regular function in $k[G]^{U(P)\times U(P^-)}$ by Theorem 3.2.8. Therefore f extends to a $U(P)$-invariant function in $\operatorname{ind}_{P^-}^G(\overline{V})$, proving surjectivity of (3.3). □

Proposition 3.3.4. *Let* $\overline{V} \in \operatorname{Rep}(M)$. *Then the following are equivalent:*

(i) $\overline{V}$ *belongs to* $\operatorname{Rep}(\overline{M})$.
(ii) The weights of $\overline{V}$ *lie in* $W_M \cdot \check{\Lambda}_G^+ \subset \check{\Lambda}$.
(iii) There exists $V \in \operatorname{Rep}(G)$ *such that* $\overline{V} \cong V^{U(P)}$.

Proof. The equivalence of (i) and (ii) follows from the definition of $\overline{M}$ in §3.1.7. Corollary 3.2.4 proves (iii) implies (ii). Lemma 3.3.3 shows that (i) implies (iii) by setting $V = \operatorname{ind}_{P^-}^G(\overline{V})$, which is a finite-dimensional G-module. □

Remark 3.3.5. Suppose that k is algebraically closed. One deduces from Lemma 3.3.3 that $\nabla(\check{\lambda})^{U(P)}$ is isomorphic to the dual Weyl M-module $\nabla_M(\check{\lambda})$. By [11, Remark II.2.11], the subspace $\nabla(\check{\lambda})^{U(P)}$ also equals the sum of the weight spaces of $\nabla(\check{\lambda})$ with weights $\leq_M \check{\lambda}$. Dually, one sees that the sum of the weight spaces of $\Delta(\check{\lambda})$ with weights $\leq_M \check{\lambda}$ is isomorphic to $\Delta(\check{\lambda})_{U(P^-)}$, which is in turn isomorphic to the Weyl M-module $\Delta_M(\check{\lambda})$.

Remark 3.3.6. Let O be a complete discrete valuation ring with field of fractions K and residue field k. By Proposition 3.3.4(iii) and the usual Tannakian formalism, one observes that the closed subscheme $\operatorname{Gr}_M^+ \subset \operatorname{Gr}_M = M(K)/M(O)$ defined in [6, §6.2], [5, §1.6] is equal to the subspace $(\overline{M}(O) \cap M(K))/M(O)$.

4. Relation to boundary degenerations

Let P and P^- be a pair of opposite parabolic subgroups in G. We identify the Levi subgroup $P \cap P^-$ with the Levi factor $M = P/U(P)$. Let $\overline{M}$ be the normal reductive monoid with group of units M defined in §3.1.7.

In this section we will show that $\overline{G/U(P)}$ embeds as a closed subscheme in the affine closure of the boundary degeneration defined in [3, 16, 15]. We will also describe the relation between the boundary degeneration and the Vinberg semigroup (i.e., enveloping semigroup) of G. This will give an alternate description of $\overline{M}$ as a subscheme of the Vinberg semigroup, using idempotents.

4.1. Boundary degenerations.

Define the boundary degeneration

$$X_P := (G \times G)/(P \underset{M}{\times} P^-) = (G/U(P) \times G/U(P^-))/(P \cap P^-),$$

where $P \cap P^-$ acts diagonally on the right. It is known that X_P is quasi-affine (cf. [7, Proposition 2.4.4]), and $k[X_P]$ is finitely generated by [10] and Hilbert's theorem on invariants. Therefore X_P is strongly quasi-affine.

Remark 4.1.1. The group $G \times G$ acts on X_P by left translations. Suppose that k is algebraically closed and choose a pair B, B^- of opposite Borel subgroups contained in P, P^-, respectively. Then the orbit of $B^- \times B$ acting on $(1,1) \in X_P$ is a dense open subset. Therefore X_P is a spherical variety with respect to $G \times G$.

4.1.2. Let $\overline{X}_P = \operatorname{Spec} k[X_P]$. Since X_P is strongly quasi-affine, $\overline{X}_P$ is affine of finite type and the canonical embedding $X_P \hookrightarrow \overline{X}_P$ is open.

Note that $\overline{X}_P$ is the affine GIT quotient of $\overline{G/U(P)} \times \overline{G/U(P^-)}$ by the diagonal right M-action, but it is *not* the stack quotient.

4.1.3. Consider the map of strongly quasi-affine varieties

$$G/U(P) \to X_P : g \mapsto (g, 1). \tag{4.1}$$

The base change of (4.1) under the smooth cover $G \times G \to X_P$ gives the natural closed embedding $G \times P^- \hookrightarrow G \times G$. Therefore (4.1) is also a closed embedding.

The composition $G/U(P) \hookrightarrow X_P \hookrightarrow \overline{X}_P$ induces a map

$$\overline{G/U(P)} \to \overline{X}_P. \tag{4.2}$$

In characteristic 0, one easily deduces from [1, Proposition 5] that (4.2) is a closed embedding. In positive characteristic, this is not *a priori* clear, but the following theorem shows it is still true:

Theorem 4.1.4. *The map* (4.2) *is a closed embedding, and the composition*

$$\overline{G/U(P)} \to \overline{X}_P \to \operatorname{Spec} k[X_P]^{U(P)}$$

is an isomorphism, where $U(P) \subset G$ acts on X_P by left translations in the second coordinate.

Proof. Observe that $k[X_P]^{U(P)} = (k[G/U(P)] \otimes k[G]^{U(P) \times U(P^-)})^M$ where M acts diagonally by right translations. Using the inversion operator on G in the second coordinate, we get $k[X_P]^{U(P)} \cong (k[G/U(P)] \otimes k[\overline{M}])^M$ where $k[\overline{M}] = k[G]^{U(P^-) \times U(P)}$ by Theorem 3.2.8 and M acts anti-diagonally on the right. Since M is dense in $\overline{M}$, the evaluation at $1 \in \overline{M}$ gives an injection $(k[G/U(P)] \otimes k[\overline{M}])^M \hookrightarrow k[G/U(P)]$. On the other hand, $\overline{M}$ is the closure of M in $\overline{G/U(P)}$ by Corollary 3.2.9. The right action of M on $G/U(P)$ therefore extends to a right action of $\overline{M}$ on $\overline{G/U(P)}$, which corresponds to a comodule map $k[G/U(P)] \to (k[G/U(P)] \otimes k[\overline{M}])^M$. The composition

$$k[G/U(P)] \to (k[G/U(P)] \otimes k[\overline{M}])^M \hookrightarrow k[G/U(P)]$$

is the identity, which proves that the composition $\overline{G/U(P)} \to \operatorname{Spec} k[X_P]^{U(P)}$ is an isomorphism. It follows that the affine map (4.2) is a closed embedding. □

Corollary 4.1.5. *Consider the embedding $M \hookrightarrow X_P$ defined as the composition of the embeddings* (3.2) *and* (4.1)*. The closure of M in $\overline{X}_P$ is isomorphic to $\overline{M}$. The composition*

$$\overline{M} \to \overline{X}_P \to \operatorname{Spec} k[X_P]^{U(P^-) \times U(P)}$$

is an isomorphism, where $U(P^-) \times U(P) \subset G \times G$ acts on X_P by left translations.

Proof. Combine Theorems 3.2.8 and 4.1.4. □

4.2. **Relation to Vinberg's semigroup.** Recall that k is an arbitrary perfect field.

4.2.1. We first give a brief review of the standard material on the Vinberg semigroup, which is contained in [18, 14, 13].

Let $Z(G)$ denote the center of G. Consider the group

$$G_{\mathrm{enh}} := (G \times T)/Z(G),$$

where $Z(G)$ maps to $G \times T$ anti-diagonally. Note that $Z(G_{\mathrm{enh}}) = T$.

The Vinberg semigroup of G, denoted $\overline{G_{\mathrm{enh}}}$, is a normal reductive k-monoid with group of units G_{enh}. The Renner cone of $\overline{G_{\mathrm{enh}}}$ is by definition

$$\{(\check{\lambda}_1, \check{\lambda}_2) \in \check{\Lambda}^{\mathbb{Q}} \times \check{\Lambda}^{\mathbb{Q}} \mid \check{\lambda}_2 - w\check{\lambda}_1 \in \check{\Lambda}_G^{\mathrm{pos},\mathbb{Q}} \text{ for all } w \in W\}, \tag{4.3}$$

where $\check{\Lambda}_G^{\mathrm{pos},\mathbb{Q}}$ is the rational polyhedral cone generated by the positive roots of G. The Vinberg semigroup may be constructed from the Renner cone as described in §2.

The canonical homomorphism of algebraic groups $G_{\mathrm{enh}} \to T_{\mathrm{adj}} := T/Z(G)$ extends to a homomorphism of algebraic monoids

$$\bar{\pi} : \overline{G_{\mathrm{enh}}} \to \overline{T_{\mathrm{adj}}},$$

where $\overline{T_{\mathrm{adj}}} := \mathfrak{t}_{\mathrm{adj}}$ is the Cartan Lie algebra of the adjoint group. Let $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ denote the non-degenerate locus of $\overline{G_{\mathrm{enh}}}$. It is known that $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ is smooth over $\overline{T_{\mathrm{adj}}}$.

4.2.2. For a parabolic P with Levi factor M, let $\mathbf{c}_P \in \overline{T_{\mathrm{adj}}}$ be the point defined by the condition that $\check{\alpha}_i(\mathbf{c}_P) = 1$ for simple roots $\check{\alpha}_i$, $i \in \Gamma_M$, and $\check{\alpha}_j(\mathbf{c}_P) = 0$ for all other simple roots. Note that $\mathbf{c}_P$ is an idempotent with respect to the monoid structure on $\overline{T_{\mathrm{adj}}}$.

Let $\overline{G_{\mathrm{enh},\mathbf{c}_P}}$ denote the fiber of $\bar{\pi}$ over $\mathbf{c}_P$. Note that by definition of $\mathbf{c}_P$, the center $Z(M)$ is the stabilizer of T acting on $\mathbf{c}_P$ in $\overline{T_{\mathrm{adj}}}$.

4.2.3. Fix a pair of opposite parabolic subgroups P and P^-, and identify M with the Levi subgroup $P \cap P^-$. Since conjugation by M fixes $Z(M)$, the center of M can be embedded as a subgroup of the abstract Cartan T. Consider the anti-diagonal map

$$\mathfrak{s} : Z(M)/Z(G) \to (Z(M) \times T)/Z(G) \hookrightarrow (G \times T)/Z(G) = G_{\mathrm{enh}}$$

defined by $\mathfrak{s}(t) = (t^{-1}, t)$. Observe that $Z(M)/Z(G) \subset T/Z(G) = T_{\mathrm{adj}}$ coincides with the subtorus $\{t \in T_{\mathrm{adj}} \mid \check{\alpha}_i(t) = 1,\, i \in \Gamma_M\}$. Let $\overline{Z(M)/Z(G)}$ denote the closure of $Z(M)/Z(G)$ in $\overline{T_{\mathrm{adj}}}$.

Lemma 4.2.4. *(i) The map $\mathfrak{s}$ extends to a homomorphism*

$$\bar{\mathfrak{s}} : \overline{Z(M)/Z(G)} \to \overline{G_{\mathrm{enh}}}$$

of algebraic monoids.

(ii) The composition $\bar{\pi} \circ \bar{\mathfrak{s}}$ is the natural inclusion $\overline{Z(M)/Z(G)} \hookrightarrow \overline{T_{\mathrm{adj}}}$.

Proof. Since we know the composition $\pi \circ \mathfrak{s}$, it suffices to prove statement (i). We may assume that k is algebraically closed.

The weight lattice of $Z(M)/Z(G)$ is the free abelian group $\check{\Lambda}_{Z(M)/Z(G)}$ with basis consisting of the simple roots $\check{\alpha}_j$ for $j \in \Gamma_G \setminus \Gamma_M$. If $\check{\lambda} = \sum_{i \in \Gamma_G} n_i \check{\alpha}_i$ for $n_i \in \mathbb{Z}$, let $\mathrm{pr}(\check{\lambda}) := \sum_{j \notin \Gamma_M} n_j \check{\alpha}_j$. Let $\check{C}$ denote the Renner cone (4.3) of $\overline{G_{\mathrm{enh}}}$, and let $\check{C}_{\mathbb{Z}} := \check{C} \cap (\check{\Lambda} \times \check{\Lambda})$. Fix a Cartan subgroup $T_{\mathrm{sub}} \subset M$ and identify T_{sub} with T by choosing a Borel. The map $\mathfrak{s}$ lands in $(T_{\mathrm{sub}} \times T)/Z(G)$, so we have an induced map of weights (restricted to $\check{C}_{\mathbb{Z}}$):

$$\check{C}_{\mathbb{Z}} \to \check{\Lambda}_{Z(M)/Z(G)} : (\check{\lambda}_1, \check{\lambda}_2) \mapsto \mathrm{pr}(\check{\lambda}_2 - \check{\lambda}_1).$$

The image of this map is the non-negative span of the simple roots $\check{\alpha}_j, j \notin \Gamma_M$. Statement (i) follows. □

Remark 4.2.5. If k is algebraically closed, then the map $\bar{\mathfrak{s}}$ we have constructed factors through the section $\overline{T_{\mathrm{adj}}} \to \overset{\circ}{\overline{G_{\mathrm{enh}}}}$ constructed in [7, Lemma D.5.2], which depends on a choice of Borel subgroup and maximal torus of G. In particular, $\bar{\mathfrak{s}}$ always lands in the non-degenerate locus of the Vinberg semigroup for arbitrary k.

4.2.6. *The idempotent e_P.* Observe that $\mathbf{c}_P$ lies in the submonoid $\overline{Z(M)/Z(G)} \subset \overline{T_{\mathrm{adj}}}$. Define the idempotent

$$e_P := \bar{\mathfrak{s}}(\mathbf{c}_P) \in \overset{\circ}{\overline{G_{\mathrm{enh}}}}(k),$$

which satisfies $\bar{\pi}(e_P) = \mathbf{c}_P$. In [7, Appendix C], it is shown (by passing to an algebraic closure $\bar{k}$) that

$$P = \{g \in G \mid g \cdot e_P = e_P \cdot g \cdot e_P\} \quad \text{and} \quad P^- = \{g \in G \mid e_P \cdot g = e_P \cdot g \cdot e_P\},$$

and the stabilizer of the $P \times P^-$ action on e_P equals $P \times_M P^-$. Note that if $g \in P \cap P^-$, then $g \cdot e_P = e_P \cdot g \cdot e_P = e_P \cdot g$. It follows that M is the centralizer of e_P in G.

Remark 4.2.7. It is known that $G \cdot e_P \cdot G$ is equal to the non-degenerate locus $\overset{\circ}{\overline{G_{\mathrm{enh},\mathbf{c}_P}}}$ of the fiber. One deduces from the above that the $G \times G$-action on e_P induces an isomorphism[6]

$$X_P := (G \times G)/(P \underset{M}{\times} P^-) \cong \overset{\circ}{\overline{G_{\mathrm{enh},\mathbf{c}_P}}}.$$

Remark 4.2.8. Suppose that k is algebraically closed. By a result of M. Putcha (cf. [13, Theorem 4.5]) for general reductive monoids, any idempotent in the non-degenerate locus of $\overline{G_{\mathrm{enh}}}$ is $G(k)$-conjugate to e_P for some parabolic P. Moreover, the choice of P and P^- determines this idempotent in its conjugacy class.

4.2.9. *Relating $\overline{M}$ to the Vinberg semigroup.* Consider the map

$$G \to \overline{G_{\mathrm{enh},\mathbf{c}_P}} : g \mapsto e_P \cdot g \cdot e_P. \tag{4.4}$$

Since $U(P) \cdot e_P = e_P \cdot U(P^-) = \{e_P\}$, this map is $U(P^-) \times U(P)$-invariant. By Theorem 3.2.8, we have an isomorphism $\overline{M} \cong \operatorname{Spec} k[G]^{U(P^-) \times U(P)}$. Since $\overline{G_{\mathrm{enh},\mathbf{c}_P}}$ is affine, the map (4.4) must factor through a map

$$\overline{M} \to e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P. \tag{4.5}$$

Observe that $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ is an irreducible algebraic monoid with identity e_P. The map (4.5) is an extension of the homomorphism of algebraic monoids $M \to e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ sending $m \mapsto m \cdot e_P = e_P \cdot m$. Therefore (4.5) must also be a homomorphism of algebraic monoids.

Theorem 4.2.10. *The homomorphism* (4.5) *is an isomorphism of algebraic monoids.*

By the definition of (4.4), we see that the image of (4.5) contains $e_P \cdot G \cdot e_P$. Since the latter map is a homomorphism of monoids, we deduce that the image contains $e_P \cdot G \cdot e_P \cdot G \cdot e_P$. By Remark 4.2.7, we have $G \cdot e_P \cdot G = \overset{\circ}{\overline{G_{\mathrm{enh},\mathbf{c}_P}}}$ is dense in $\overline{G_{\mathrm{enh},\mathbf{c}_P}}$. Multiplying on the left and right by e_P, we deduce that (4.5) has dense image. On the other hand, the restriction of (4.5) to M is injective. It follows that $M \cdot e_P$ is a dense sub*group* of $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$. Therefore $M \cdot e_P$ must be equal to the group of units of $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$.

[6]In fact, we learned from S. Schieder that this induces an isomorphism of affine varieties $\overline{X}_P \cong \overline{G_{\mathrm{enh},\mathbf{c}_P}}$.

We show that the monoid $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ is normal and then use Renner's classification of normal monoids to prove the theorem.

Consider the larger algebraic monoid $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ with unit e_P (where we do not restrict to a fiber). The action of $Z(G_{\mathrm{enh}}) = T$ on $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ induces an isomorphism

$$((e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P) \times T)/Z(M) \cong e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P, \tag{4.6}$$

so the two aforementioned monoids are closely related.

Since $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ is the closed subscheme of the Vinberg semigroup fixed by left and right multiplications by e_P, it is a retract of $\overline{G_{\mathrm{enh}}}$ in the category of schemes. The retraction is given by the formula $x \mapsto e_P \cdot x \cdot e_P$.

Lemma 4.2.11. *Let Y and S be integral affine schemes such that Y is a retract of S (i.e., there exist maps $Y \to S$ and $S \to Y$ such that their composition is the identity map on Y). If S is normal then so is Y.*

Proof. Since Y is a retract of S, we have an inclusion of algebras $k[Y] \hookrightarrow k[S]$. The algebra $k[S]$ is integrally closed, so if $\tilde{Y}$ denotes the normalization of Y in its field of fractions, then the previous inclusion factors as $k[Y] \hookrightarrow k[\tilde{Y}] \hookrightarrow k[S]$. On the other hand the map $Y \to S$ induces an algebra map $k[S] \to k[Y]$ which restricts to the identity on $k[Y]$. Localization implies that the composition $k[\tilde{Y}] \to k[Y]$ is injective and hence an isomorphism. □

Corollary 4.2.12. *The algebraic monoid $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ is normal.*

Proof. The Vinberg semigroup is normal by definition, and we have observed that $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ is a retract of $\overline{G_{\mathrm{enh}}}$. □

Corollary 4.2.13. *The algebraic monoid $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ is normal.*

Proof. We deduce from (4.6) that $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ is smooth locally isomorphic to $(e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P) \times T$. It follows from Corollary 4.2.12 and ascending and descending properties of normality that $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ is normal. □

Proof of Theorem 4.2.10. By Corollary 4.2.13 we know that $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ is a normal reductive monoid with group of units $M \cdot e_P$. Recall from §2 that normal reductive monoids are classified by their Renner cones. Since $\overline{M}$ is also a normal reductive monoid with group of units M, to prove the theorem it suffices to check that the Renner cones of $\overline{M}$ and $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ are equal. We may assume that k is algebraically closed.

Fix a Cartan subgroup $T_{\mathrm{sub}} \subset M \subset G$. Identify T_{sub} with the abstract Cartan T by choosing a Borel subgroup. Consider the embedding $T_{\mathrm{sub}} \hookrightarrow \overline{G_{\mathrm{enh}}}$ sending $t \mapsto t \cdot e_P$ and let $\overline{T_{\mathrm{sub}} \cdot e_P}$ denote the closure of the image. Set $T_{\mathrm{enh}} := (T_{\mathrm{sub}} \times T)/Z(G)$, which is a Cartan subgroup of G_{enh}, and let $\overline{T_{\mathrm{enh}}}$ denote its closure in $\overline{G_{\mathrm{enh}}}$. By definition, e_P lies in $\overline{T_{\mathrm{enh}}}$, so $T_{\mathrm{sub}} \hookrightarrow \overline{G_{\mathrm{enh}}}$ factors through the homomorphism of monoids

$$T_{\mathrm{sub}} \hookrightarrow \overline{T_{\mathrm{enh}}} : t \mapsto t \cdot e_P \tag{4.7}$$

Let $\check{C} \subset \check{\Lambda}^{\mathbb{Q}} \times \check{\Lambda}^{\mathbb{Q}}$ denote the Renner cone (4.3) of $\overline{G_{\mathrm{enh}}}$. Recall that the weights in $\check{C}_{\mathbb{Z}} := \check{C} \cap (\check{\Lambda} \times \check{\Lambda})$ form a basis of $k[\overline{T_{\mathrm{enh}}}]$. Let $(\check{\lambda}_1, \check{\lambda}_2) \in \check{C}_{\mathbb{Z}}$. Then $\check{\lambda}_2 - \check{\lambda}_1 \in \check{\Lambda}_G^{\mathrm{pos}}$, so it may be considered as a regular function on $\overline{T_{\mathrm{adj}}}$. Evaluating this function at $\mathbf{c}_P$ gives a number $(\check{\lambda}_2 - \check{\lambda}_1)(\mathbf{c}_P)$, which is 1 if $\check{\lambda}_2 - \check{\lambda}_1 \in \check{\Lambda}_M^{\mathrm{pos}}$ and 0 otherwise. By the definition of e_P, one sees that the homomorphism (4.7) corresponds to the map of weights

$$\check{C}_{\mathbb{Z}} \to \check{\Lambda} : (\check{\lambda}_1, \check{\lambda}_2) \mapsto (\check{\lambda}_2 - \check{\lambda}_1)(\mathbf{c}_P) \cdot \check{\lambda}_1. \tag{4.8}$$

The existence of the map (4.5) implies that the image of (4.8) must land in the Renner cone of $\overline{M}$, which is generated by the saturated submonoid $W_M \cdot \check{\Lambda}^+_G$. On the other hand, for $\check{\lambda} \in \check{\Lambda}^+_G$ and $w \in W_M$, one sees that $(w\check{\lambda}, \check{\lambda}) \mapsto w\check{\lambda}$. Thus the image of (4.8) equals $W_M \cdot \check{\Lambda}^+_G$.

Therefore the Renner cones of $\overline{M}$ and $e_P \cdot \overline{G_{\mathrm{enh},\mathbf{c}_P}} \cdot e_P$ are equal, which proves the theorem. □

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