SMOOTH NON-ADMISSIBLE ASYMPTOTICS FOR $\mathrm{SL}_2(\mathbb{R})$

JONATHAN WANG

Abstract. The goal of this note is to demonstrate that a notion of smooth asymptotics map exists without K-finiteness assumptions for $\mathrm{SL}_2(\mathbb{R})$ using explicit known formulas for intertwining operators. This suggests that a theory parallel to Bernstein's p-adic theory, further developed by Bezrukavnikov–Kazhdan [BK], should exist for real groups as well.

Preliminaries

Let $G = \mathrm{SL}_2(\mathbb{R})$. Let B denote the minimal parabolic of upper triangular matrices. We have the Langlands decomposition $B = NAM^1$ where N is the unipotent radical of B, we identify A with $\mathbb{R}_+^\times = (0,\infty)$ via $a \mapsto \left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right)$ and $M^1 = \{\pm 1\}$. We have $M = AM^1 = \mathbb{R}^\times$ is a Levi subgroup of B. Let $K = \mathrm{SO}(2)$ denote the maximal compact subgroup of G.

Identify $N^-\backslash G$ with $V \setminus 0$ where $V = \mathbb{R}^2$ and $N\backslash G$ with $V^* \setminus 0$. Under this identification $\left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right) \in N^-\backslash G$ identifies with $\left(\begin{smallmatrix} a^{-1} & \\ & a\end{smallmatrix}\right) e_2 = ae_2$ and $\left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right) \in N\backslash G$ identifies with $e_2^* \left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right) = a^{-1}e_2^*$.

So if we want to define $\|\bar{n}amk\| = \|a\|$ for $\bar{n}amk \in N^-\backslash G$, this identifies with the usual norm $\|v\|$ on $v \in V \setminus 0$. On the other hand $\|namk\| = \|a\|$ for $namk \in N\backslash G$ corresponds to $\|\xi\|^{-1}$ on $\xi \in V^* \setminus 0$.

0.1. Let $X = M^{diag}\backslash(N^-\backslash G \times N\backslash G)$, which identifies with the space of rank one 2×2 matrices. Then the main result, intended to parallel [BK, Proposition 7.1, Theorem 7.6], can be interpreted as:

Theorem 0.1.1 (cf. Theorem 3.4.3)**.** *There exists a G-equivariant map*

$$\mathrm{Asymp} : \mathscr{S}(G) \to \widehat{C}_+(X)$$

that recovers asymptotics of matrix coefficients, where $\mathscr{S}(G)$ is Casselman's Schwartz space and $\widehat{C}_+(X)$ is some "completed" space of formal series.

1. The spaces $\mathscr{S}_\pm^{\mathrm{umg}}$

We have the algebraic Schwartz space $\mathscr{S}(N^-\backslash G)$ which roughly consists of functions f on $N^-\backslash G = V \setminus 0$ such that all derivatives are rapidly decreasing as $\|v\| \to 0$ and ∞. This is a Fréchet space and SF-module over $\mathscr{S}(M)$ and $\mathscr{S}(G)$ (cf. [CH, 4.3, 4.4]).

In our case, we have $N^-\backslash G = A \times K = A \times S^1$. We can consider $f \in C^\infty(N^-\backslash G)$ as a function $f(a,\theta)$ where $a = \|v\|$ and θ is the angular variable. We will also use polar coordinates to consider a function $f \in C^\infty(N\backslash G) = C^\infty(V^* \setminus 0)$ as a function $f(a,\theta)$ where $a = \|\xi\|$. We caution that the A-action by left translation corresponds to scaling by a on $N^-\backslash G = V \setminus 0$ but to scaling by a^{-1} on $N\backslash G = V^* \setminus 0$.

In these coordinates, the semi-norms on $\mathscr{S}(N^-\backslash G)$ are of the form

$$\|f\|_{\alpha,\beta,r} := \sup_{a,\theta} |(a\partial_a)^\alpha \partial_\theta^\beta f| \cdot a^r$$

where α, β are non-negative integers and $r \in \mathbb{R}$.

We now introduce the space $\mathscr{S}_+^r(N^-\backslash G)$ for $r \in \mathbb{R}$. This space consists of all smooth functions $f : N^-\backslash G = V \setminus 0 \to \mathbb{C}$ such that

$$\|f\|_{+,\alpha,\beta,R} := \sup_{a\geq 1,\theta} |(a\partial_a)^\alpha \partial_\theta^\beta f| \cdot a^R < \infty$$

for all $R \in \mathbb{R}$ and all $\alpha, \beta \geq 0$, and

$$\|f\|_{\alpha,\beta,r} < \infty$$

for our fixed number r and all $\alpha, \beta \geq 0$. Then $\mathscr{S}_+^r(N^-\backslash G)$ becomes a Fréchet space with respect to these semi-norms.

We made the definition so that $U\mathfrak{a} = \mathbb{C}[a\partial_a]$ acts on $\mathscr{S}_+^r(N^-\backslash G)$ by left translations. One can check that ∂_a sends $\mathscr{S}_+^r(N^-\backslash G) \to \mathscr{S}_+^{r+1}(N^-\backslash G)$.

Define the LF-space

$$\mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G) = \operatorname*{colim}_{r\in\mathbb{R}_+} \mathscr{S}_+^r(N^-\backslash G).$$

This is the space of smooth functions f such that all derivatives are rapidly decreasing as $\|v\| \to \infty$ and f has *uniform moderate growth*[1] as $\|v\| \to 0$.

Analogously define $\mathscr{S}_-^{\mathrm{umg}}(N^-\backslash G)$ with the two directions flipped.

We also define the spaces $\mathscr{S}_\pm^{\mathrm{umg}}(N\backslash G)$ using these definitions where $\|v\|$ is replaced by $\|\xi\|$ for $\xi \in V^* \setminus 0 = N\backslash G$.

2. Intertwining operator R

We have the standard intertwining operator

$$R = R_B : C_c^\infty(N^-\backslash G) \to C^\infty(N\backslash G)$$

defined by $Rf(g) = \int_N f(ng)dn$. Recall that with our identifications this equals

$$Rf(\xi) = \int_{\langle \xi, v\rangle = 1} f(v) d\mu_\xi.$$

Evidently R is right G-equivariant and hence $U\mathfrak{g}$-equivariant.

We have that R is almost equivariant with respect to the left M-action in the sense that

$$\check{\alpha}(a) \cdot Rf = a^2 R(\check{\alpha}(a) \cdot f),$$

where $\check{\alpha}(a) \cdot f(g) := f(\check{\alpha}(a)g)$. Note that $\check{\alpha}(a)$ acts on both $N^-\backslash G = V \setminus 0$ by the scalar a but on $N\backslash G = V^* \setminus 0$ by the scalar a^{-1}. We deduce that in polar coordinates,

$$-a\partial_a(Rf) = 2Rf + R(a\partial_a f) \tag{2.1}$$

Lemma 2.0.1. *The operator R extends to a continuous operator $\mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G) \to \mathscr{S}_-^{\mathrm{umg}}(N\backslash G)$. More specifically, R sends $\mathscr{S}_+^r(N^-\backslash G) \to \mathscr{S}_-^{-r+2}(N\backslash G)$.*

Proof. Using the relation (2.1) and induction, it suffices to check the bounds on semi-norms for $\alpha = \beta = 0$. For the purposes of bounding semi-norms we can replace f by the radial function

$$F(a) = \sup_{\|x\|=a} |f(x)|.$$

If $f \in \mathscr{S}_+^r(N^-\backslash G)$ then $F(a) = O(a^{-r})$ as $a \to 0$ and $F(a)$ is rapidly decreasing as $a \to \infty$. Then

$$|Rf(\xi)| \leq \|\xi\|^{-1} \int_{\mathbb{R}} F(\sqrt{\|\xi\|^{-2} + x^2}) dx.$$

[1]Here uniform is with respect to $U\mathfrak{a}$. I am not sure this definition is natural.

Since $F(\sqrt{\|\xi\|^{-2}+x^2})$ is rapidly decreasing as $x\to\infty$, the integral converges. If $\|\xi\|\geq 1$ then $\int_0^\infty F(\sqrt{\|\xi\|^{-2}+x^2})dx$ is bounded by a constant times

$$\int_0^1(\|\xi\|^{-2}+x^2)^{-r/2}dx=\|\xi\|^{r-1}\int_0^{\|\xi\|^{-1}}(1+x^2)^{-r/2}dx\leq\|\xi\|^{r-1}\int_0^1(1+x^2)^{-r/2}dx$$

where $\int_0^1(1+x^2)^{-r/2}dx$ is just a finite number. This shows that $|Rf(\xi)|=O(\|\xi\|^{r-2})$ as $\|\xi\|\to\infty$.

If $\|\xi\|\leq 1$ then $\int_0^\infty F(\sqrt{\|\xi\|^{-2}+x^2})dx$ is dominated by a constant multiple of

$$\int_1^\infty(\|\xi\|^{-2}+x^2)^{-R/2}dx=\|\xi\|^R\int_0^\infty(1+(\|\xi\|x)^2)^{-R/2}dx\leq\|\xi\|^R\int_0^\infty(1+x^2)^{-R/2}dx$$

for all R. This shows that $Rf\in\mathscr{S}_-^{-r+2}(N\backslash G)$.

A careful check of the constant multipliers shows continuity with respect to the semi-norms.

□

2.1. Decomposition into K-types. Since $K=\mathrm{SO}(2)$, the K-types are 1-dimensional and indexed by $\mathbb{Z}$.

For any $f\in\mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G)$ we have a Fourier series (i.e., infinite K-type decomposition)

$$f(a,\theta)=\sum_{n\in\mathbb{Z}}f_n(a)e^{in\theta}$$

where $f_n(a)$ is a function on $A=\mathbb{R}_+^\times$ which is uniform moderate growth at 0 and all derivatives rapidly decreasing at ∞. The sum (and all its derivatives) converges absolutely since

$$(in)^\beta f_n=\frac{1}{2\pi}\int_0^{2\pi}f(a,\theta)\cdot(-\partial_\theta)^\beta(e^{-in\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}\partial_\theta^\beta f\cdot e^{-in\theta}d\theta$$

implies that

$$|n^\beta f_n(a)|\leq\sup_\theta|\partial_\theta^\beta f(a,\theta)|$$

for all $\beta\in\mathbb{N}$. Using the same reasoning (e.g., for $\beta+2$ above), we see that

$$\|f\|_{\alpha,\beta,r}=\sup_{a,\theta}|(a\partial_a)^\alpha\partial_\theta^\beta f|\cdot a^r\leq\sum_{n\in\mathbb{Z}}|n|^\beta\sup_a\{|(a\partial_a)^\alpha f_n|\cdot a^r\}\leq 2\zeta(2)\|f\|_{\alpha,\beta+2,r}<\infty \tag{2.2}$$

when $f\in\mathscr{S}_+^r(N^-\backslash G)$. Similarly, we have

$$\|f\|_{+,\alpha,\beta,R}\leq\sum_{n\in\mathbb{Z}}n^\beta\sup_{a\geq 1}\{|(a\partial_a)^\alpha f_n|\cdot a^R\}\leq 2\zeta(2)\|f\|_{+,\alpha,\beta+2,r}<\infty$$

and analogous bounds for the semi-norms on $\mathscr{S}_\pm^{\mathrm{umg}}(N^\pm\backslash G)$.

2.2. Mellin transform and c-function. For a fixed K-type $n\in\mathbb{Z}$, consider $f_n(a)e^{in\theta}\in\mathscr{S}_+^r(N^-\backslash G)$. Then we can define the Mellin transform of f_n by

$$\hat{f}_n(z)=\int_0^\infty f_n(a)a^{z-1}da$$

where the RHS is absolutely convergent for $z\in\mathbb{C}$ with $\mathrm{Re}(z)>r$. Mellin inversion theorem says that

$$f_n(a)=\frac{1}{2\pi i}\int_{-\infty}^\infty a^{-c-iy}\hat{f}_n(c+iy)dy$$

for $c>r$. Note that for $c>r$, we have $f_n(a)a^c\in L^2(A,\frac{da}{a})$.

We know how the intertwining operator acts on principal series, cf. [W, Lemma 7.17]. If we abuse notation, this says that

$$R(a^{-z}e^{in\theta}) = c_n(z)a^{z-2}e^{in\theta} \tag{2.3}$$

where R is defined by the same integral as before, which converges absolutely for $\mathrm{Re}(z) > 1$, and we have a formula

$$c_n(z) = \frac{\pi^{1/2}\Gamma(\frac{z-1}{2})\Gamma(\frac{z}{2})}{\Gamma(\frac{z+n}{2})\Gamma(\frac{z-n}{2})}. \tag{2.4}$$

Since all integrals converge absolutely, by Fubini and Mellin inversion we have

$$R(f_n(a)e^{in\theta}) = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=c} \hat{f}_n(z)c_n(z)a^{z-2}dz \cdot e^{in\theta}$$

If we return to a general $f \in \mathscr{S}^r_+(N^-\backslash G)$ then we can express Rf as the sum of the RHS above over all $n \in \mathbb{Z}$ since the Fourier series for Rf is absolutely convergent.

3. Description of the inverse

3.1. The K-finite setting. First we fix a K-type $n \in \mathbb{Z}$ and take $f_n(a)e^{in\theta} \in \mathscr{S}^{-r}_-(N\backslash G)$. Recall that this implies f_n is rapidly decreasing as $a \to 0$ and $f_n(a) = O(a^r)$ as $a \to \infty$.

Then the Mellin transform $\hat{f}_n(z)$ is absolutely convergent for $\mathrm{Re}(z) < -r$. Note that for $c < -r$, we have $f_n(a)a^c \in L^2(A, \frac{da}{a})$ and consequently $\hat{f}_n(z) \in L^2(c + i\mathbb{R})$.

Motivated by (2.3), we want to define

$$R'(f_n(a)e^{in\theta}) = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=c} \hat{f}_n(z)c_n(-z+2)^{-1}a^{z-2}dz \cdot e^{in\theta} \tag{3.1}$$

for $c \ll 0$.

Remark 3.1.1. The difference between this definition and the one in [CH] is that in *loc cit.* they integrate over $\mathrm{Re}(z) = 1$. This is the same distinction as between smooth and L^2 asymptotics in the non-Archimedean case.

Now we use the explicit formula (2.4) for c_n to make estimates. A consequence of Stirling's approximation is that

$$\lim_{|z|\to\infty} \frac{\Gamma(z+a)}{\Gamma(z)}z^{-a} = 1, \quad |\arg z| \le \pi - \varepsilon$$

for any $a \in \mathbb{C}$ and $\varepsilon > 0$. Using this, we get the asymptotic approximation

$$c_n(z) \sim \pi^{1/2}(\tfrac{z}{2})^{-1/2}. \tag{3.2}$$

if $\arg(z - |n|) \le \pi - \varepsilon$

We deduce that $c_n(-z+2)^{-1} \sim \pi^{-1/2}(\frac{-z}{2})^{1/2}$ in any vertical line as $\mathrm{Im}(z) \to \infty$.

Remark 3.1.2. The key observation to note is that this approximation *is independent of the K-type n*. This is a general phenomenon for any G, cf. [VW, Lemma 3.5].

Lemma 3.1.3. *The integral*

$$R'_{n,c}(f_n)(a) := \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=c} \hat{f}_n(z)c_n(-z+2)^{-1}a^{z-2}dz$$

converges absolutely if $f_n e^{in\theta} \in \mathscr{S}^{-r}_-(N\backslash G)$ and $c < -r$ and $c \notin -n+2+\mathbb{Z}_+$. More specifically, we have

$$|R'_{n,c}(f_n)(a)| \le \tilde{B}_c(f_n)a^{c-2} \tag{3.3}$$

where $\tilde{B}_c(f_n)$ is a linear combination of semi-norms on f_n with positive coefficients depending on c but not on n.

Proof. Rewrite

$$\int_{\mathrm{Re}(z)=c} |\hat{f}_n(z)c_n(-z+2)^{-1}|dz = \int_{\mathrm{Re}(z)=c} \frac{|c_n(-z+2)|^{-1}}{|(z+1)z|} \cdot |(z+1)z\hat{f}_n(z)|dz.$$

The $(z+1)z$ in the denominator is to ensure that $\frac{|c_n(-z+2)|^{-1}}{|(z+1)z|} \in L^2(c+i\mathbb{R})$. Integration by parts implies that $(\partial_a^2 f)^\wedge(z+2) = (z+1)z\hat{f}(z)$.

If $f_n \in \mathscr{S}_-^{-r}(N\backslash G)$, then $\partial_a^2 f_n \in \mathscr{S}_-^{-r+2}(N\backslash G)$. Thus $(\partial_a^2 f_n)^\wedge(z+2)$ converges absolutely for $\mathrm{Re}(z) < -r$ and $(\partial_a^2 f_n)^\wedge(z+2) \in L^2(c+i\mathbb{R})$. Now the Cauchy–Schwartz inequality gives the bound

$$\int_{\mathrm{Re}(z)=c} |\hat{f}_n(z)c_n(-z+2)^{-1}|dz \leq \left\| \frac{c_n(-z+2)^{-1}}{(z+1)z} \right\|_{L^2(c+i\mathbb{R})} \cdot \|(\partial_a^2 f_n)^\wedge\|_{L^2(c+2+i\mathbb{R})}.$$

Here $\|\frac{c_n(-z+2)^{-1}}{(z+1)z}\|_{L^2(c+i\mathbb{R})}$ can be bounded above by a constant B_c independent of n but dependent on c. By Plancherel theorem,

$$\|(\partial_a^2 f_n)^\wedge\|^2_{L^2(c+2+i\mathbb{R})} = \int_A |\partial_a^2 f_n(a)a^{c+2}|^2 \frac{da}{a}.$$

Since $f_n \in \mathscr{S}_-^{-r}(N\backslash G)$, the RHS is bounded above by

$$(\|f_n\|_{1,0,-r} + \|f_n\|_{2,0,-r})^2 \int_1^\infty a^{2(c+r)-1}da + (\|f_n\|_{-,1,0,R} + \|f_n\|_{-,2,0,R})^2 \int_0^1 a^{2(c-R)-1}da < \infty,$$

where $R \in \mathbb{R}$ can be taken arbitrarily negative. ☐

Lemma 3.1.3 shows that the integral defining $R'_{n,c}$ converges absolutely on all derivatives of $f_n(a)$ as well, so the dominated convergence theorem implies that $R'(f_n(a)e^{in\theta})$ is a smooth function on $N^-\backslash G$. So we get an operator

$$R'_{n,c} : \mathscr{S}_-^{-r}(N\backslash G)_n \to C^\infty(N^-\backslash G)_n,$$

where the subscript n denotes the K-isotypic component corresponding to $n \in \mathbb{Z}$.

By dominated convergence theorem, we see that

$$a\partial_a R'(f_n(a)e^{in\theta})) = \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=c} \hat{f}_n(z)(z-2)c_n(-z+2)^{-1}a^{z-2}dz \cdot e^{in\theta}$$

where $(z-2)\hat{f}_n(z) = -(a\partial_a f_n)^\wedge(z) - 2\hat{f}_n(z)$. We deduce that

$$a\partial_a R'(f_n e^{in\theta}) = -R'(a\partial_a f_n e^{in\theta}) - 2R'(f_n e^{in\theta}). \tag{3.4}$$

We can therefore inductively get similar bounds as (3.3) for all the derivatives $|(a\partial_a)^\alpha R'(f_n e^{in\theta})|$.

By consideration of the poles of $c_n(z)^{-1}$, one observes that $R'_{n,c}$ is independent of c as long as $c < \min\{-|n|+2, -r\}$ by contour shift.

Corollary 3.1.4. *We have a continuous operator*

$$R' = R'_{n,c} : \mathscr{S}_-^{-r}(N\backslash G)_n \to \mathscr{S}_+^{\max\{r+2,|n|\}+\varepsilon}(N^-\backslash G)_n$$

independent of $c < \min\{-|n|+2, -r\}$, where $\varepsilon > 0$ is arbitrary.

Proof. For such a c, the bound (3.3) implies that all $\mathfrak{a}$-derivatives of $R'_{n,c}(f_n)$ grow like $O(a^{c-2})$ as $a \to 0$. Since we can move c arbitrarily far to the left, we also see that $R'_{n,c}(f_n) = O(a^{-R})$ as $a \to \infty$ for all $R \in \mathbb{R}$. ☐

We can easily extend R' to an operator

$$R' : \mathscr{S}_-^{\mathrm{umg}}(N\backslash G)_{(K)} \to \mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G)_{(K)}$$

where the subscript (K) denotes the K-finite functions. We have essentially constructed R' so that the following is true:

Lemma 3.1.5. *Let $f_n e^{in\theta} \in \mathscr{S}_-^{\mathrm{umg}}(N\backslash G)_n$. Then*

$$(R'(f_n e^{in\theta}))^\wedge(z) = \hat{f}_n(-z+2)c_n(z)^{-1} \tag{3.5}$$

for $\mathrm{Re}(z) \gg 0$.

Proof. Since $R'(f_n e^{in\theta})$ is defined by the absolutely convergent integral (3.1), which is essentially an inverse Mellin transform, the claim follows by the usual (L^1) Fourier inversion theorem. □

By Mellin inversion theorem we get the following corollary:

Corollary 3.1.6. *The operator*

$$R : \mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G)_{(K)} \to \mathscr{S}_-^{\mathrm{umg}}(N\backslash G)_{(K)}$$

is a topological isomorphism with inverse given by R'.

3.2. The problem. The naive attempt would be to extend R' to an operator $\mathscr{S}_-^{\mathrm{umg}}(N\backslash G) \to \mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G)$ without K-finiteness by taking Fourier series. Unfortunately, this has a bug: in order to make R' well-defined, we had to shift move c to the left so that $c < -|n| + 2$. If we sum over all n, we would need to move c infinitely far to the left, so there is vertical line one can integrate over to define R'.

3.3. A theorem of Schwartz. Without K-finiteness, we do have the following inversion theorem, due to Schwartz himself: we have an embedding

$$\mathscr{S}(\mathbb{R}^2) \hookrightarrow \mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G).$$

We also have an embedding

$$\mathscr{S}(S^1 \times \mathbb{R}) \hookrightarrow \mathscr{S}_-^{\mathrm{umg}}(N\backslash G)$$

where $F \in \mathscr{S}(S^1 \times \mathbb{R})$ is sent to $f(a\omega) = a^{-1}F(\omega, a^{-1})$ for $\omega = (\omega_1, \omega_2) \in S^1$. Let $\mathscr{S}_H(S^1 \times \mathbb{R})$ denote the subspace of functions F such that

$$\int_{\mathbb{R}} F(\omega, a)a^m da = \int_{\mathbb{R}} f(a\omega)a^{-m-1} da =: \hat{f}(-m)(\omega)$$

is a homogeneous polynomial in ω_1, ω_2 of degree m for each $m \in \mathbb{Z}_+$. Note that this implies $\hat{f}(-m) \in \mathscr{S}(K)$ has K-type $\pm m$.

Theorem 3.3.1 (Schwartz, cf. [H, Theorem 2.4])**.** *The operator R defines an isomorphism*

$$\mathscr{S}(\mathbb{R}^2) \xrightarrow{\sim} \mathscr{S}_H(S^1 \times \mathbb{R}).$$

It is also perhaps worth pointing out that the inverse is defined by first taking a 1-dimensional Fourier transform [2] with respect to $\|\xi\|$ and then taking 2-dimensional Fourier transform to get a function in $\mathscr{S}(\mathbb{R}^2)$.

[2] This Fourier transform is really a *multiplicative* convolution $*r^{-1}e^{-ir\|\xi\|}dr$ of the function in $\mathscr{S}_-^{\mathrm{umg}}(N\backslash G)$, which Ngo would call a kind of Hankel transform

3.4. Removing K-finiteness. Let us continue our approach. If we have $f_n e^{in\theta} \in \mathscr{S}_-^{-r}(N\backslash G)_n$, we can use residue theorem to shift the contour, picking up residues along the way: if we fix a small $\varepsilon > 0$, then

$$R'_{n,c}(f_n) = R'_{n,-r-\varepsilon}(f_n) - \sum_{m>r, m\in\mathbb{Z}} \hat{f}_n(-m)\left(\mathrm{Res}_{z=m+2}\, c_n(z)^{-1}\right) \cdot a^{-m-2} \tag{3.6}$$

where $c_n(z)^{-1}$ only has a pole at $m+2$ if $m \leq n-2$, so the sum is finite in the K-finite setting (but will become infinite if we remove K-finiteness).

As a first step, we can use Lemma 3.1.3 to take Fourier series of $R'_{n,-r-\varepsilon}$ to get an operator on all of $\mathscr{S}_-^{-r}(N\backslash G)$.

Theorem 3.4.1. *Let $f \in \mathscr{S}_-^{-r}(N\backslash G)$. For a fixed $\varepsilon > 0$, the sum*

$$R'_{-r-\varepsilon}(f)(a,\theta) := \sum_{n\in\mathbb{Z}} R'_{n,-r-\varepsilon}(f_n)(a)e^{in\theta}$$

converges absolutely and defines an operator

$$R'_{-r-\varepsilon} : \mathscr{S}_-^{-r}(N\backslash G) \to C^\infty(N^-\backslash G)$$

with the property that $\|R'f\|_{\alpha,\beta,r+2+\varepsilon} < \infty$ for all α,β, and $R'_{-r-\varepsilon}$ is continuous with respect to this semi-norm.

Proof. For $f \in \mathscr{S}_-^{-r}(N\backslash G)$ not necessarily K-finite, we have the Fourier series $f(a,\theta) = \sum_{n\in\mathbb{Z}} f_n(a)e^{in\theta}$. Let $r' = r+2+\varepsilon$. Applying (2.2), we have the bound

$$\|R'_{-r-\varepsilon} f\|_{0,\beta,r'} \leq \sum_{n\in\mathbb{Z}} |n|^\beta \sup_a \{|R'(f_n(a)e^{in\theta})|a^{r'}\}$$

Using (3.3), the RHS is bounded by

$$\sum_{n\in\mathbb{Z}} |n|^\beta \tilde{B}_c(f_n).$$

Recall that $\tilde{B}_c(f_n)$ is a linear combination of semi-norms $\|f_n\|_{\alpha,0,r'}$ where the coefficients do not depend on n. The second inequality in (2.2) implies that

$$\sum_{n\in\mathbb{Z}} |n|^\beta \|f_n\|_{\alpha,0,r'} \leq 2\zeta(2)\cdot \|f\|_{\alpha,\beta+2,r'}.$$

Therefore we conclude that

$$\|R'_{-r-\varepsilon} f\|_{0,\beta,r'} \leq \tilde{B}_c(f) < \infty$$

where $\tilde{B}_c(f)$ is a positive linear combination of semi-norms $\|f\|_{\alpha,\beta,r'}$. Using (3.4), we can inductively get similar bounds for $\|R'_{-r-\varepsilon} f\|_{\alpha,\beta,r'}$ for all $\alpha,\beta \geq 0$. The exact same argument gives continuous bounds on $\|R'_{-r-\varepsilon} f\|_{+,\alpha,\beta,R}$ for any $R \in \mathbb{R}$. This proves the theorem. □

Now for fixed $m \in \mathbb{Z}$ let us look at the series

$$R_m^\sharp(f)(\theta) := \sum_{n\in\mathbb{Z}} \hat{f}_n(-m)e^{in\theta}\left(\mathrm{Res}_{z=m+2}\, c_n(z)^{-1}\right) \tag{3.7}$$

Assuming $m > -2$, from (2.4) we get

$$\mathrm{Res}_{z=m+2}\, c_n(z)^{-1} = \frac{-(-1)^{\frac{n-m}{2}} 2^{m+1} (\frac{m+n}{2})!}{\pi(\frac{n-m}{2}-1)!m!} \quad \text{if } n \in m+2+2\mathbb{Z}_+$$

and vanishes otherwise. By Stirling's approximation again, this has magnitude asymptotically equal to $\frac{1}{\pi\cdot m!} n^{m+1}$ as $|n| \to \infty$.

For $f \in \mathscr{S}_-^{-r}(N\backslash G)$, we can consider the Mellin transform $\hat{f}(z)$ as a meromorphic function with values in $\mathscr{S}(K) = \mathscr{S}(S^1)$ via the usual integral

$$\hat{f}(z)(\theta) = \int_0^\infty f(a,\theta) a^{z-1} da$$

for $\mathrm{Re}(z) < -r$. Then $\hat{f}_n(-m)$ is the n-th Fourier coefficient of $\hat{f}(-m) \in \mathscr{S}(K)$. As such, $\hat{f}_n(-m)$ decreases more rapidly than any polynomial in n. Consequently, the series (3.7) converges absolutely and so do all its derivatives. Therefore $R_m^\sharp(f)$ defines a smooth function on $K = S^1$.

Now we can go back and take the Fourier series of (3.6) to at least formally write

$$R'(f)(a,\theta) = R'_{-r-\varepsilon}(f) - \sum_{m>r, m\in\mathbb{Z}} R_m^\sharp(f)(\theta) a^{-m-2} \tag{3.8}$$

If f has no K-types n with $|n| \geq m+2$, then $R_m^\sharp(f) = 0$.

Remark 3.4.2. If f lies in the image of $\mathscr{S}_H(S^1 \times \mathbb{R}) \hookrightarrow \mathscr{S}_-^{\mathrm{umg}}(N\backslash G)$ as in the setting of Theorem 3.3.1, then $\hat{f}_n(-m) = 0$ unless $m = |n|$, so all of the $R_m^\sharp(f) = 0$.

Question: What space does the right hand side of (3.8) live in?

We consider $R'(f)$ as some kind of formal series of functions (or perhaps more accurately distributions) on $N^-\backslash G = V \setminus 0 = \mathbb{R}_+^\times \times S^1 = A \times K$. Then the right hand side of (3.8) can be considered as an element of

$$\widehat{C}_+(N^-\backslash G) := C^\infty(N^-\backslash G) \underset{\mathscr{S}(K)[a^{-1}]}{\otimes} \mathscr{S}(K)[\![a^{-1}]\!].$$

By residue theorem, (3.8) is independent of $\varepsilon > 0$ when considered as an element of $\widehat{C}_+(N^-\backslash G)$. Since $\mathscr{S}_-^{\mathrm{umg}}(N\backslash G)$ is a colimit, we deduce:

Theorem 3.4.3. *The formula* (3.8) *defines an operator*

$$R' : \mathscr{S}_-^{\mathrm{umg}}(N\backslash G) \to \widehat{C}_+(N\backslash G).$$

4. Relation to asymptotics

In this section, which is entirely expository, we return to the K-finite situation where things are better understood. The goal is simply to spell out the relation between the inverse intertwining operator and asymptotics of functions on G (which is probably well-known to experts). In other words, we want to verify an archimedean version of [BK, Theorem 7.6]. At the heart, this is about the relation between the c-function, μ-function, and Plancherel theorem, which all originate with Harish-Chandra.

Fix a two sided irreducible representation τ of K, i.e., two integers n, m corresponding to K-types. We will only consider τ-spherical functions in this section.

We will follow the notation of [BK]. Let $X = M^{diag}\backslash(N^-\backslash G \times N\backslash G)$, the space of rank 1 2×2 matrices. Let $Y = M^{diag}\backslash(N\backslash G \times N\backslash G)$.

Recall that the right G-invariant measure on $N\backslash G$ with respect to the NMK-decomposition is given by $\delta(m)^{-1} dndmdk = |m|^{-2} dndmdk$. Here dm is the multiplicative Haar measure on $M = \mathbb{R}^\times$. We will use the notation $dm = \frac{da}{a}$ where da stands for the additive Haar measure on $\mathbb{R}$.

4.1. Principal series and Fourier transform. Let $\mathscr{M}(Y)$ be the space of smooth functions[3] on $N\backslash G \times N\backslash G$ that satisfy $f(my_1, y_2) = \delta(m)f(y_1, m^{-1}y_2)$ that are Schwartz functions modulo M (on Y). We have the action map

$$\mathsf{A} : \mathscr{S}(G) \to \mathscr{M}(Y), \qquad \mathsf{A}(f)(y_1, y_2) = \int_N f(y_2^{-1} n y_1) dn, \qquad y_1, y_2 \in G.$$

Note this is $G \times G$ equivariant if we define the action on $\mathscr{S}(G)$ by $((g_1, g_2)f)(x) = f(g_2^{-1} x g_1)$. Elements of $\mathscr{M}(Y)$ can be considered as operators on $C^\infty(N\backslash G)$. In this sense A does correspond to the action of $\mathscr{S}(G)$ on $C^\infty(N\backslash G)$. For $\psi \in C^\infty(N\backslash G)$, we have

$$\int_{N\backslash G} \psi(y_1)\mathsf{A}(f)(y_1, y_2) dy_1 = \int_G \psi(y_2 g) f(g) dg.$$

Meanwhile, the Fourier transform of $\mathscr{S}(G)$ in the sense of [A] comes from considering the action of $\mathscr{S}(G)$ on the principal series. So *the Mellin transform of* A *is the Fourier transform*, which we will make precise below.

Let us fix our notation for normalized principal series to agree with [A, C]. Let σ be either the trivial or sign representation of $M/A = \{\pm 1\}$. If ind_B^G denotes the un-normalized induction, then for $z \in \mathbb{C}$ let

$$I_B^G(\sigma \cdot a^z) = \mathrm{ind}_B^G(\sigma \cdot a^{z+1}) = \{f \in C^\infty(N\backslash G) \mid f(nmg) = \sigma(m)|m|^{z+1} f(g),\, n \in N, m \in M\},$$

where $\delta^{1/2}(m) = |m|$ and we are identifying M with $\mathbb{R}^\times$ via $\check{\alpha}$. Then $I_B^G(\sigma \cdot a^z)$ is a smooth, admissible G-representation, and its contragredient is isomorphic to

$$I_B^G(\sigma \cdot a^z)^\sim \cong I_B^G(\sigma \cdot a^{-z})$$

and the pairing is given by

$$I_B^G(\sigma \cdot a^z) \times I_B^G(\sigma \cdot a^{-z}) \to \mathbb{C} : \langle \varphi, f \rangle = \int_{B\backslash G} f\varphi dg.$$

Note that $I_B^G(\sigma \cdot a^z)$ only has K-types n such that $\sigma(-1) = (-1)^n$.

Now if $f \in \mathscr{S}(G)_\tau$, then $\mathsf{A}(f) \in \mathscr{M}(Y)_{n,m}$ and $\mathsf{A}(f)$ is determined by the restriction to $A \times 1$. Define

$$\hat{\mathsf{A}}(f)(z) = \int_0^\infty \mathsf{A}(f)(a, 1) a^{z-1} da,$$

which converges for all $z \in \mathbb{C}$. Then $\hat{\mathsf{A}}(f)(z)$ corresponds to the action of f on $I_B^G(\sigma \cdot a^{z+1})$ where $\sigma(-1) = (-1)^n = (-1)^m$.

4.2. Eisenstein integrals. Assume from now on that $\sigma(-1) = (-1)^n = (-1)^m$. We can identify $I_B^G(\sigma \cdot a^z) = \mathrm{ind}_{K_M}^K(\sigma|_{K_M}) = \text{c-ind}_{K_M}^K(\sigma|_{K_M})$. Let $V_n = \mathbb{C}e^{in\theta}$ denote the 1-dimensional representation of K corresponding to K-type n. By Frobenius reciprocity,

$$\mathrm{Hom}_K(V_n, I_B^G(\sigma \cdot a^z)) = \mathrm{Hom}_{K_M}(V_n, \sigma) = \mathbb{C}.$$

The map corresponding to 1 sends $1 \cdot e^{in\theta} \in V_n$ to the function $\varphi_n \in I_B^G(\sigma \cdot a^z)$ with

$$\varphi_n(mk_\theta) = \mathrm{sign}(m)^n |m|^{z+1} e^{in\theta}, \quad m \in M = \mathbb{R}^\times,\, k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

[3]Really $\mathscr{M}(Y)$ denotes sections of the analytic line bundle of measures on fibers of $Y \to B\backslash G$ via second projection. There is no $G \times G$-invariant measure on Y, so the identification with smooth functions is not canonical.

We similarly have a function $\tilde{\varphi}_m \in I_B^G(\sigma \cdot a^{-z})_m$. Now the matrix coefficient $\langle \tilde{\varphi}_m, g \cdot \varphi_n \rangle \in C^\infty(G)$ corresponds to the image of $e^{in\theta} \otimes e^{im\theta}$ under the composition

$$\tau \to I_B^G(\sigma \cdot a^z) \otimes I_B^G(\sigma \cdot a^z)^\sim \to C^\infty(G)$$

and is given by the integral

$$\langle \tilde{\varphi}_m, g \cdot \varphi_n \rangle = \int_{B\backslash G} (g \cdot \varphi_n)\tilde{\varphi}_m = \int_K \varphi_n(k_\theta g) e^{im\theta} dk_\theta =: E_B(g,z)$$

where K has measure 1. The integral $E_B(g,z)$ agrees with the usual notion of *Eisenstein integral* due to Harish-Chandra (the integral depends on τ and σ; it is simpler in the case $G = \mathrm{SL}_2(\mathbb{R})$ because τ and σ are both 1-dimensional).

We can consider the dual of A as an operator $\mathsf{A}^* : \mathscr{M}'(Y) \to \mathscr{S}'(G)$. Fixing choices of Haar measure, we identify distributions with generalized functions. Let τ_M denote $\tau|_{K_M}$. Restricting to τ-spherical distributions, we have

$$\mathsf{A}^* : \mathscr{S}'(A) = \mathscr{S}'(M)_{\tau_M} = \mathscr{M}'(Y)_\tau \to \mathscr{S}'(G)_\tau.$$

The key observation that helps to relate [BK] to the classical literature on harmonic analysis is that *the Mellin transform of* A^* *is the Eisenstein integral.*

For $f \in \mathscr{S}(G)_{\tau^*}$ and $\phi \in \mathscr{M}(Y)_\tau$, we have[4]

$$\begin{aligned}
\langle \mathsf{A}^*(\phi), f \rangle = \langle \phi, \mathsf{A}(f) \rangle &= \int_Y \phi(y)\mathsf{A}(f)(y)dy \\
&= \int_{K\times K} \int_A \mathsf{A}(f)(ak_1,k_2)\phi(ak_1,k_2)\delta(a)^{-1}\frac{da}{a}dk_1 dk_2 \\
&= \int_{K\times K}\int_A\int_N \tau(1,k_2) f(k_2^{-1} n a k_1)\phi(ak_1,1)\delta(a)^{-1} dn \frac{da}{a} dk_1 dk_2 \\
&= \int_G f(g) \left(\int_K \phi(k_\theta g, 1) e^{im\theta} dk_\theta \right) dg
\end{aligned}$$

From these equalities and Mellin inversion we see that

$$\hat{\mathsf{A}}^*(\phi)(g,z) := \int_0^\infty \mathsf{A}^*(\phi)(a\cdot g, 1) a^{z-1} da = E_B(g, -z-1).$$

This gives another proof that $\hat{\mathsf{A}}$ is the Fourier transform, since Fourier transform is by definition the adjoint of the Eisenstein integrals. [5]

4.3. Asymptotics map. We can define $\mathscr{S}_+^{\mathrm{umg}}(X)$ analogously to $\mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G)$. In fact I believe that

$$\mathscr{S}_+^{\mathrm{umg}}(X) \cong \mathscr{S}_+^{\mathrm{umg}}(N^-\backslash G) \mathop{\widehat{\otimes}}_{\mathscr{S}(M)} \mathscr{S}(N\backslash G).$$

We can also define $\mathscr{M}_-^{\mathrm{umg}}(Y)$ analogously to $\mathscr{S}_-^{\mathrm{umg}}(N\backslash G)$. Again I believe we have an identification

$$\mathscr{M}_-^{\mathrm{umg}}(Y) \cong (\mathscr{S}_-^{\mathrm{umg}}(N\backslash G) \mathop{\widehat{\otimes}}_{\mathscr{S}(M)} \mathscr{S}(N\backslash G)) \mathop{\widehat{\otimes}}_{\mathscr{S}(Y)} \mathscr{M}(Y).$$

Now looking at just the $K \times K$-finite part, Corollary 3.1.6 implies that we have a topological isomorphism

$$R^{-1} \otimes 1 : \mathscr{M}_-^{\mathrm{umg}}(Y)_{(K\times K)} \xrightarrow{\sim} \mathscr{S}_+^{\mathrm{umg}}(X)_{(K\times K)}.$$

---

[4]We have a pairing between $\mathscr{M}(Y)$ and itself by integrating on Y.

[5]Informally: $\mathsf{A}^*(a^{-z+1}) = E_B(g,-z)$. Then $\langle f, E_B(g,-z)\rangle = \langle f, \mathsf{A}^*(a^{-z+1})\rangle = \langle \mathsf{A}(f), a^{-z+1}\rangle_Y = \int_A \mathsf{A}(f)(a,1)a^{-z+1}\delta(a)^{-1}da$.

Using [BK, Theorem 7.6] as *motivation*, we can define a $G \times G$-equivariant operator

$$\mathrm{Asymp} : \mathscr{S}(G)_{(K\times K)} \to \mathscr{S}_+^{\mathrm{umg}}(X)_{(K\times K)}$$

as the composition

$$\mathscr{S}(G)_{(K\times K)} \xrightarrow{\mathsf{A}} \mathscr{M}(Y)_{(K\times K)} \hookrightarrow \mathscr{M}_-^{\mathrm{umg}}(Y)_{(K\times K)} \xrightarrow{R^{-1}\otimes 1} \mathscr{S}_+^{\mathrm{umg}}(X)_{(K\times K)}.$$

The operator Asymp is denoted by B^* in [BK], and it is the dual of the smooth Bernstein map. In the non-archimedean case, Asymp actually extends to an operator $C^\infty(G) \to C^\infty(X)$ so it makes sense to evaluate it on matrix coefficients of smooth G-representations. In the archimedean case, Asymp does not extend: equivalently, the dual operator B does not send $\mathscr{S}(X)$ to $\mathscr{S}(G)$ (if it did, this would give a proof of second adjointness, which definitely does not hold in the archimedean setting). The failure is related to the infinite number of poles of the c-function, which is what the previous sections tried to explain.

In the non-archimedean setting, the unique characterization of Asymp is that it is $G \times G$-equivariant and for a τ-spherical function f, we have

$$\mathrm{Asymp}(f)(1, a) = f(a) \qquad \text{for} \quad |a| \ll 1,$$

cf. [BK, Lemma 5.5].

In the archimedean setting, we can no longer expect a true equality, so the analogous statement we will check as an analog of [BK, Theorem 7.6] is that $\mathrm{Asymp}(f)$ and f indeed have the same asymptotic behavior as $a \to 0$.

Proposition 4.3.1. *Let $f \in \mathscr{S}(G)_\tau$. Then*

$$\mathrm{Asymp}(f)(1, a) \sim f(a)$$

as $a \to 0$, where $\sim$ means that the limit of the ratio goes to 1.

The idea is that the asymptotics of Eisenstein integrals (i.e., matrix coefficients of principal series) are given by the c-function. This goes back to Langlands and Harish-Chandra. Then using a result of Arthur, we express any τ-spherical Schwartz function as an integral of Eisenstein integrals. The result of Arthur is sophisticated, but I believe I am using an easy part of it.

Proof. By a continuity argument, one should be able to assume $f \in C_c^\infty(G)_\tau$. Now we use some facts from the proof of the main theorem in [A]. Recall that τ corresponds to integers n, m. We assume $\sigma(-1) = (-1)^n = (-1)^m$ and suppress σ from the notation.

In [A, III.2, p. 73] the Fourier transform is defined by

$$F(z) = \int_G f(g) E_B(g, -z) dg, \qquad z \in \mathbb{C}.$$

In our notation, $F(z) = \hat{\mathsf{A}}(f)(-z-1)$ corresponds to the action of f on $I_B^G(\sigma \cdot a^{-z})$.

Then the proof of Arthur shows (cf. [A, p. 4]) that

$$f(a) = \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=c} \mu_\tau(z) F(z) E_{B|B,1}(a, z) dz \tag{4.1}$$

for $a \in A, a < 1$ and $c \ll 0$. Here μ_τ is Harish-Chandra's μ-function. (The hard part of [A] was showing that this identity held even at $a = 1$.) There is a decomposition

$$E_B(a, z) = E_{B|B,1}(a, z) + E_{B|B,w_0}(a, z)$$

for $a < 1$ which is uniquely determined by an asymptotic expansion which we now recall (cf. [A, I.4]). Recall that $E_B(a,z) = \langle \tilde{\varphi}_m, a \cdot \varphi_n \rangle$ is the matrix coefficient of the principal series. By a classical argument of Langlands (cf. [C, Theorem 13.1]), we have

$$\langle \tilde{\varphi}_m, a \cdot \varphi_n \rangle \sim a^{1-z} \int_N \varphi_n(w_0 n) dn$$

as $a \to 0$ for $\mathrm{Re}(z) > 0$. In our notation, $\int_N \varphi_n(w_0 n) dn = R(a^{-(z+1)} e^{in\theta})(1) = c_n(z+1)$ so

$$E_B(a,z) \sim c_n(z+1) a^{1-z}$$

as $a \to 0$. Then $E_{B|B,1}(a,z)$ is defined to have asymptotic approximation

$$E_{B|B,1}(a,z) \sim c_n(z+1) a^{1-z}$$

as $a \to 0$, for any $z \in \mathbb{C}$.

Above $\mu_\tau(z)$ denotes the Harish-Chandra μ-function with normalization incorporated. In our notation, $\mu_\tau(z) = (c_n(z+1)c_n(-z+1))^{-1} = (c_m(z+1)c_m(-z+1))^{-1}$ under the assumption $(-1)^n = (-1)^m$. Combined with (4.1), we deduce that

$$f(a) \sim \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=c} F(z) c_n(1-z)^{-1} a^{1-z} dz. \tag{4.2}$$

On the other hand, by Mellin inversion,

$$\mathsf{A}(f)(a^{-1}, 1) = \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=c-1} \hat{\mathsf{A}}(f)(-z) a^{-z} dz$$

We want to apply R^{-1} in the first variable. Recall that under the identification $N\backslash G = V^* \setminus 0$, the action of $\alpha(a)$ scales V^* by a^{-1}. Thus we can apply (3.1) to get

$$\begin{aligned} \mathrm{Asymp}(f)(a,1) &= \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=c-1} \hat{\mathsf{A}}(f)(-z) c_n(-z+2)^{-1} a^{z-2} dz \\ &= \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=c} F(z) c_n(1-z)^{-1} a^{z-1} dz. \end{aligned}$$

Comparing with (4.2), we conclude that $f(a)$ has the same asymptotics as $\mathrm{Asymp}(f)(a^{-1},1) = \mathrm{Asymp}(f)(1,a)$ as $a \to 0$. □

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