Construction of $\check{G}_X$

Want reductive group $\check{G}_X/\mathbb{C}$ with $\check{G}_X \times SL_2 \longrightarrow \check{G}$ group homo

$\varphi : \check{G}_X \longrightarrow \check{G}$ has finite kernel

- $\check{G}_X$ defined by [SV], existence of φ under assumptions about $\underbrace{\check{G}_{X,GN} \subset \check{G}}$ Tannakian defn by Galtsgory–Nadler
- Knop–Schalke define $\check{G}_X, \varphi$ for any G-variety X
  - Combinatorial

In spirit: [GN] : $\underbrace{\mathrm{Perv}(LX/L^+G)}_{\text{doesn't exist}} \cong \mathrm{Rep}(\check{G}_X)$ LX = Loop space, L^+G = Arc space

[Sakellaridis]: $C_c^\infty(X(F_v))^{G(\mathcal{O}_v)} \cong \mathbb{C}[\check{G}_X]^{\check{G}_X}$ in some cases

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$X = H\backslash G$ homogeneous, quasi-affine, spherical variety over k (alg closed, char 0)

We give overview of Knop–Schalke:

Based root datum of G:

$(\Lambda, \Delta_G, \Lambda^\vee, \check{\Delta}_G)$ Λ: weight lattice, Δ_G: simple roots (wrt B)

Homogeneous spherical varieties classified by Luna, Bravi–Pezzini

homogeneous spherical datum:

$(\Lambda_X, \Sigma_X, \mathcal{D}, c: \mathcal{D} \to \check{\Lambda}_X, M \subset \mathcal{D} \times \Delta_G)$

$\Lambda_X \subset \Lambda$

Σ_X = spherical roots (simple)

$\mathcal{D}$ = colors := prime B-divisors

$k(X)^{(B)}$ = B-eigenspaces (X spherical $\Rightarrow$ mult one)

$1 \to k^\times \to k(X)^{(B)} \to \Lambda_X \to 1$

Λ_X = set of B-eigenvalues in $k(X)$ ~~fraction~~ field.

$\Lambda_X \hookrightarrow \Lambda_G \Rightarrow$ $\check{\Lambda}_G^{\mathbb{Q}} \twoheadrightarrow \check{\Lambda}_X^{\mathbb{Q}}$

$\check{\Lambda}_G \to \check{\Lambda}_X$ has finite cokernel $\rightsquigarrow$ $\check{A}_X \downarrow \check{T}$ finite kernel $T \downarrow A_X$

$\mathcal{V} :=$ rational cone of G-invariant valuations on $k(X)$.

$\boxed{\text{Thm [GN] 8.2.9} \quad X(F_v)/G(\mathcal{O}_v) = \{\text{discrete } G\text{-valuations of } k(X)\}}$

$\mathcal{V} \longrightarrow \check{\Lambda}_X^{\mathbb{Q}}$ by restriction to $k(X)^{(B)}$. Fact: this map is injective (Knop)

$\mathcal{V}$ used in Luna–Vust theory of spherical embeddings

$\mathcal{V}$ contains image of $-\check{\Lambda}^{+,\mathbb{Q}}_G$ under $\check{\Lambda}^{\mathbb{Q}}_G \twoheadrightarrow \check{\Lambda}^{\mathbb{Q}}_X$

equality $\overset{\text{def}}{\iff}$ wave front

$\underbrace{\mathcal{V}}_{\text{first defined by Brion}}$ is fundamental domain for $\underbrace{W_X}_{\text{little Weyl group of Knop}} \hookrightarrow W$ } this is probably most geometric defn of $\mathcal{V}$

" $= -\check{\Lambda}^{+,\mathbb{Q}}_X$ "

Knop defines action of W_X on B-orbits of X ('95)

$\mathcal{V}^\vee_{\mathbb{R}} := \{ \lambda \in \Lambda^{\mathbb{R}}_X \mid \langle \lambda, v\rangle \le 0 \quad \forall v \in \mathcal{V}\}$

$\Sigma_X :=$ generators of intersections of extremal rays of $\mathcal{V}^\vee_{\mathbb{R}}$ with Λ_X.

Alternate defn: rational cone gen'd by Σ_X
= cone gen'd by ν s.t. $k[X]_\lambda \cdot k[X]_\mu \xrightarrow{\text{project}} k[X]_{\lambda+\mu-\nu}$ is nonzero

Fact Σ_X are linearly independent, called spherical roots (but in fact simple roots of a root system)

$c : \mathcal{D} \longrightarrow \check{\Lambda}_X$ given by valuation defined by a color

$M \subset \mathcal{D} \times \Delta_G$ $\quad M = \{ (D, \alpha) \mid D$ is "unstable" under $P_\alpha \}$

$\mathring{X}$ = dense B-orbit

$P(X) = \{ g \in G \mid \mathring{X} g = \mathring{X} \}$ $\quad L(X) =$ Levi

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Weak spherical datum:

$(\Xi, \Delta_X, \Delta_{L(X)})$

Fact: $\sigma \in \Sigma_X$, there is $c \in \{1, 2, \frac{1}{2}\}$ s.t. $\sigma_{norm} := c\sigma$ is equal to either a root of G or $\alpha + \beta$ where α, β roots of G

$(\mathbb{Q}\alpha + \mathbb{Q}\beta) \cap \Lambda_G = \{\pm\alpha, \pm\beta\}$

there's a canonical way to decompose $\sigma_{norm} = \gamma_1 + \gamma_2$ (associated roots (positive))

$\Delta_X := \{ \sigma_{norm} \mid \sigma \in \Sigma_X \}$ (renormalize lengths of roots so $\check{\gamma}$ lies in $\check{\Lambda}_X$)

$\Xi := \Lambda_X + \mathbb{Z}\Delta_X$

There are ~~basically~~ 3 types of normalized spherical roots: $\sigma \in \Delta_X$

Type T: $\sigma \in \Phi_G$ and $\sigma \in \Lambda_X$

$X = \mathbb{G}_m \backslash PGL_2 \qquad (\check{G}_X = SL_2 = \check{G})$

Type G: $\sigma = \gamma_1 + \gamma_2$. Always have $\sigma \in \Lambda_X$ in this case

$X = PGL_2 \backslash PGL_2 \times PGL_2 \qquad \sigma = (\alpha, \alpha)$

Type N: $\sigma \in \Phi_G$ but $\sigma \notin \Lambda_X$ ($2\sigma \in \Lambda_X$)

$X = N(T) \backslash PGL_2 = PO_2 \backslash PGL_2$ We want

$$\begin{array}{ccc} \check{A}_X & \hookrightarrow & \widetilde{SL_2} \\ \downarrow 2 & & \downarrow 2 \\ \check{A} & \hookrightarrow & SL_2 = \check{G} \end{array}$$

$\leftarrow$ not algebraic

Knop–Schalke construct $\check{G}_X$ attached to $(\Xi, \Delta_X, \Delta_{L(X)})$.

but torus of $\check{G}_X$ corresponds to $\Xi \supset \Lambda_X$

equals $\check{A}_X$ if $\Xi = \Lambda_X \iff X$ has no spherical roots of "type N".

$O_n \backslash GL_n$ is bad case mentioned in [SV]).

They construct $\check{G}_X \longrightarrow \check{G_X^\wedge} \subset \check{G}$ by "folding"

where $G_X^\wedge$ has torus $\check{T}$ and roots are $\check{\sigma}$, $\underbrace{\check{\gamma}_1, \check{\gamma}_2}_{\text{associated to } \sigma = \gamma_1 + \gamma_2}$

Then show existence of $\check{G}_X \times SL_2 \longrightarrow \check{G}$ homomorphism where

$SL_2 \longrightarrow \check{G}$ is the principal SL_2 for $\check{L}(X)$ (dual gp of $L(X)$). $\check{L}(X) \subset \check{G}$

using classification of rank 2 spherical varieties.

Examples

Thm The spherical subgroups $H \subset G = PGL_2$ $(k = \bar{k}$, char $0)$ are

$\Lambda_G = \mathbb{Z}\alpha$

	X_H		
• Type G:	$PGL_2 \backslash PGL_2$	$\check{G}_X = \{1\}$	
• Type T:	$T \backslash PGL_2$	$\check{G}_X = \check{G} = SL_2$	three B-orbits: one open, two colors (divisors)
• Type U:	$SU \backslash PGL_2$, $S \subset N_G(U)$, U unipotent	$\check{G}_X = (\check{T}/S)$ ($\Leftrightarrow X$ horospherical)	two B-orbits, one open, one closed (color)
• Type N:	$N(T) \backslash PGL_2 = PO_2 \backslash PGL_2$	$\Lambda_X = \mathbb{Z}(2\alpha) \subset \Lambda_G$, $\Delta_X = \{\alpha\} \notin \Lambda_X$, $\check{A}_X \xrightarrow{2} \check{T}$	two B-orbits, one open, one closed of smaller rank

Other examples

• $X = H$ $\quad G = H \times H \hookleftarrow H^{\text{diag}}$

$\check{G}_X = \check{H}$

$\Phi_X = \{ (\alpha, -w_0^H \alpha) \mid \alpha \in \Phi_H \} \subset \Lambda_H \times \Lambda_H = \Lambda_G$

$\Phi_G = \{ (\alpha, 0), (0, \alpha) \} = \Phi_H \times \Phi_H \subset \Lambda_H \times \Lambda_H = \Lambda_G$

• $X = \dfrac{(PGL_2)^3}{PGL_2}$ $\quad \check{G}_X = \check{G}$ $\quad$ five B orbits: one open, three divisors, one closed

$X/B = PGL_2 \backslash (\mathbb{P}^1)^3$