Topics in Calculus and Algebra

Taught by: I. Grojnowski

<u>Lecture 1</u> 01/19/12

E_d-algebras, (∞, d)-cat's, applications...

Background reading for simplicial sets: Goerss & Schemmerhorn, arxiv/0609537

Next lecture: (∞, d)-categories

<u>Next 3(?) lectures</u>: explaining Lurie's statement of theorem about TFT and (∞, d)-cats

Classical d-TFT, Atiyah, after Segal

<u>Def</u> d-Cob (for Cobordism) category: objects: closed, oriented manifolds of dim $d-1$

mor: $\{$bordisms from M to $N\}$/diffeo, rel ∂

i.e. B is a oriented d-dim mfld equipped w/ an orientation preserving diffeo $\partial B \simeq \overline{M} \amalg N$

$\overline{M} = M$ with opp. orientation

Consider B & B' same if $\exists$ diffeo $B \xrightarrow{\varphi} B'$ extending given diffeos

$\partial B' \xrightarrow{\sim} \overline{M} \amalg N \xleftarrow{\sim} \partial B$

E.g. $d=2$ 2-Cob

Objects: closed 1-mflds 

as any connected closed orient. mfld is diffeo & cobordant to $S^1 = O$

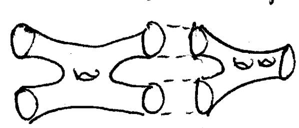
ob 2-Cob $\simeq \mathbb{N} = \{0, 1, \ldots\}$ disjoint union of circles

 $\leftarrow$ example of cobordism between $O\,O$ and $O\,O$

Identity cobord is $M \times I$

<u>thm / observation / fudge</u>:

d-Cob is a category, i.e. you can compose cobordisms



Need to prove glued space has str. of a smooth manifold, indep of classes, depending only on diffeo type at ∂

// the details of this will be filled in via course

Moreover, d-Cob is a symmetric monoidal, wrt. ~~disjoint~~ disjoint union

i.e. M, N manifolds, so is $M \amalg N$ $\quad M \amalg N \simeq N \amalg M$

ϕ = empty mfld, $\phi \amalg M = M \amalg \phi = M$, associativity ...

Def Atiyah, after Segal

a d-TFT is a sym-monoidal functor $Z : (d\text{-Cob}, \amalg) \longrightarrow (\mathrm{Vec}, \otimes)$

$\mathrm{Vec} = $ v.s$/k$, k a field

ie. Z functor $d\text{-Cob} \longrightarrow \mathrm{Vec}$

+ isos $\varphi_{M,N} : Z(M \amalg N) \xrightarrow{\sim} Z(M) \otimes Z(N)$ compatible w/ symmetry, assoc, etc

E.g. $Z(\phi \amalg M) = Z(\phi) \otimes Z(M)$

$\quad \overset{\shortparallel}{Z(M)} \Rightarrow Z(\phi) = k$

If $\dim M = d-1$, $Z(M)$ a vs$/k$

M  N $\xrightarrow{Z}$ linear map

B

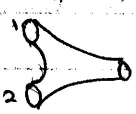
$Z(B) : Z(M) \longrightarrow Z(N)$

If $\partial B = \phi$, then $Z(B) : \underset{k}{Z(\phi)} \longrightarrow \underset{k}{Z(\phi)}$ linear map, so $Z(B) \in k$

$\boxed{d=2}$ (1) Any 1-mfld is cobord to disjoint union of circles.

$Z(S^1 \amalg \dots \amalg S^1) = Z(S^1) \otimes \dots \otimes Z(S^1)$

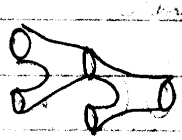
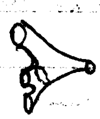
so $Z(\text{1-mfld}) = Z(S^1)^{\otimes \underbrace{\pi_0(\text{1-mfld})}_{\#\text{ of comp.}}}$ $\qquad Z(S^1) = A \in \mathrm{Vec}$

Consider $B =$  $Z(B) = m : A \otimes A \longrightarrow A$ linear, call it "multiplication"

Now B is diffeo to same thing w/ 1, 2 swapped.

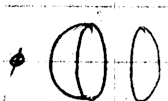
so I can swap 1 & 2 $\Rightarrow$ m is commutative $\quad A\otimes A \xrightarrow{\text{swap}} A \otimes A \xrightarrow{m} A$

$m(a_1 \otimes a_2) = m(a_2 \otimes a_1)$

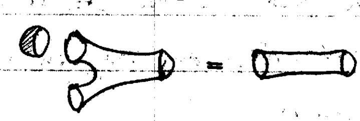
  gives associativity

$$\begin{tikzcd} (A\otimes A)\otimes A \arrow[r, phantom, "\simeq"] \arrow[d, "m\otimes 1"] & A\otimes(A\otimes A) \arrow[d, "1\otimes m"] \\ A\otimes A \arrow[dr, "m"] & A\otimes A \arrow[d] \\ & A \end{tikzcd}$$

//All subtleties will be dealt w/ next week. Best way to deal w/ it is thm of Lurie

Unit:  $Z(\text{◯}) : Z(\phi) = k \longrightarrow Z(S^1) = A$

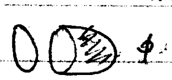
$k \longrightarrow A \longleftrightarrow$ "1" $\in A$

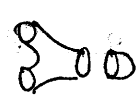
 ie. $A \simeq k \otimes A \xrightarrow{\text{coev}\otimes 1} A \otimes A \xrightarrow{m} A$

is the identity.

ie. "1" is unit for mult.

Same disk w/ op orientation:

 $Z(\text{◯}) : Z(S^1) \longrightarrow Z(\phi)$

$\mathrm{tr} : A \longrightarrow k$ 

Exercise Show $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, so get sym, bilinear form $A \otimes A \longrightarrow k$

$a \otimes b \longmapsto \mathrm{tr}(ab)$

subtle point: bilinear form is NON-DEGENERATE

In particular, A is $\underline{\text{f.d.}}$

Prop $Z : d\text{-Cob} \longrightarrow \mathrm{Vec}$ a d-TFT

then $Z(M)$ is $\underline{\text{f.d.}}$ and $Z(\bar{M}) = Z(M)^*$ is dual v.s.

$\forall$ closed $(d-1)$ manifolds M

Pf/explanation) if V vs. $/k$, $V^* = \mathrm{Hom}_k(V,k)$ dual v.s.

have $ev : V \otimes V^* \longrightarrow k$ $\quad v \otimes \varphi \mapsto \varphi(v)$

If V is f.d., also have $coev : k \longrightarrow V^* \otimes V$ $\quad 1 \mapsto \sum e_i^* \otimes e_i$

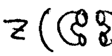
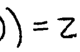
where e_i, e_i^* dual bases of V, V^*

(note: always have $k \to \mathrm{Hom}(V,V)$, but $\mathrm{Hom}(V,W) = V^* \otimes W$ only if V is f.d.)

$1 \mapsto \mathrm{Id}$

(So for example $k \xrightarrow{coev} V^* \otimes V \xrightarrow{swap} V \otimes V^* \longrightarrow k$

$1 \longmapsto \sum e_i^* \otimes e_i \longrightarrow \dim V$)

i.e. for $d=2$, $Z($  $) = Z($  $) = \dim A$

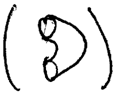
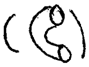
these satisfy compat. $V = V \otimes k \xrightarrow{1 \otimes ev} V \otimes V^* \otimes V \xrightarrow{ev \otimes 1} k \otimes V = V$

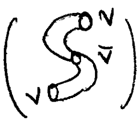
$v = v \otimes 1 \longmapsto \sum v \otimes e_i^* \otimes e_i \longmapsto \sum e_i^*(v) e_i = v$

& similarly, $V^* = k \otimes V^* \xrightarrow{coev \otimes 1} V^* \otimes V \otimes V^* \xrightarrow{1 \otimes ev} V^*$

are identity

Put $V = Z(M)$, $\bar{V} = Z(\bar{M})$

$Z($  $) : V \otimes \bar{V} \longrightarrow k$, $\quad Z($  $) :$ $k \longrightarrow V \otimes \bar{V}$

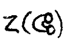
$Z($  $)$ $\Rightarrow$ $V \longrightarrow V \otimes \bar{V} \otimes V \longrightarrow V$ is identity

& similarly $\bar{V} \to \bar{V} \otimes V \otimes \bar{V} \to \bar{V}$ identity

$V^*, \bar{V}$ both inverses, but inverses unique

pairing $\bar{V} \otimes V \to k$ gives rise to map $\bar{V} \longrightarrow V^*$

$\bar{v} \longmapsto (x \in V \mapsto ev(\bar{v} \otimes x))$

but $V^* = V^* \otimes k \xrightarrow{1 \otimes Z(\;)}$  $V^* \otimes V \otimes \bar{V} \xrightarrow{ev \otimes 1} k \otimes \bar{V} \xrightarrow{\sim} \bar{V}$

Exercise the resulting map $V^* \longrightarrow \bar{V}$ is inverse of $\bar{V} \longrightarrow V^*$. $\square$

i.e. $Z \in 2\text{-TFT} \longmapsto A$ a f.d. comm. alg $/k$ w/

- $tr : A \to k$ st. $tr(ab)$ $\underline{\text{non-degen.}}$ symm bilinear form

"Frobenius algebra"

Thm ("folk") Conversely, given A a Frob alg, get $Z \in 2\text{-TFT}$, uniquely.

Remark Any 2-manifold can be cut up into pieces that are diffeo

add to generator either  or 

~~But~~ So Z of these determine Z

But you can cut these up in multiple ways, so point is that all of these relations are consequences of the ones we wrote

[Try getting $Z($$)$]

E.g. $A = \mathbb{C}[x]/x^n$, $tr(f)$ = coeff of x^{n-1} : $A \to \mathbb{C}$ determines a 2-TFT

E.g. $\mathbb{C}[G]$, G a finite gp

Exercise $Z($$) = ?$ $|G|^g$

Exercise Show 1-TFT $\longleftrightarrow$ f.d. v.s. V

$Z(\cdot) = V$ $\quad Z(S^1) = \dim V$

dimensional reduction:

if $Z : (d+1)\text{-Cob} \to \text{Vec}$ is $(d+1)$TFT

then $Z(\cdot \times S^1) : d\text{-Cob} \to \text{Vec}$ is a d-TFT.

Eg. Z a 2-TFT $Z(S^1) = A$, a Frob alg is a f.d. v-s.

hence if Z is a 3-TFT, then $Z(S^1 \times S^1) = A$ is a Frob alg.

but $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, so $SL_2\mathbb{Z}$ acts on $S^1 \times S^1$ & on A.

Baez-Dolan cobord hyp: extend a d-TFT down to a point

d-manifold $\rightsquigarrow \lambda \in k$, ts a v.s.

$(d-1)$ $\rightsquigarrow V$ a vs. $\in$ Vect, a $\mathbb{C}$-linear

$(d-2)$ $\rightsquigarrow$ Vect, an object in a 2-cat

$\vdots$

pt 0-dim $\rightsquigarrow \mathcal{C}$ a d-Cat

extended d-TFT is a sym monoidal functor F: "d-cat of d-cobord" $\to$ some d-Cat

they conjectured · you can make sense of this

· F is completely determined by $F(\cdot)$

for $d=2$, already not obv. ← you want chain complexes, not vect.

thm of Kontsevich, Costello

[What is (∞, d)-cat?]

for general d, Lurie

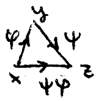
Lecture 2 01/24/12

Refs. 0901.3602 . Article we will follow for next few lectures

$\mathcal{C}$ cat $\rightsquigarrow$ top space $|N\mathcal{C}|$

ob $\mathcal{C}$ $\rightsquigarrow$ points

$\varphi \in \mathcal{C}(X,Y) \to$ interval with endpoints X, Y

$x \xrightarrow{\varphi} y \xrightarrow{\psi} z \longrightarrow$ 2 simplex  σ_2 with boundary as shown

$$\sigma_n = \{x \in \mathbb{R}^{n+1} \mid \sum_0^n x_i = 1\}$$

this factors into a combinatorial part $\mathcal{C} \to N\mathcal{C}$ simplicial set

& a "realisation part" $|\cdot| : \Delta^{op}\mathrm{Set} \to \mathrm{Top}$

def of $\Delta^{op}\mathrm{Set}$, Nerve N, ...

Δ = cat of finite ordered sets

equiv to cat with ob $\Delta = \{[0],[1],\dots\}$, $[n] = \{0,1,2,\dots,n\}$

$\mathcal{C}$ any cat, $\Delta^{op}\mathcal{C} = s\mathcal{C} = PSh(\Delta^{op}, \mathcal{C})$ = functors $\Delta^{op} \to \mathcal{C}$

"simplicial $\mathcal{C}$ objects"

$X \in \Delta^{op}\mathcal{C}$, write $X_n = X([n])$

$d_i : X_n \to X_{n-1}$ for $X(d^i)$ "face maps" $d^i : [n-1] \to [n]$ skip i

$s_i : X_{n-1} \to X_n$ for $X(s^i)$ "degeneracies" $s^i : [n] \to [n-1]$ double up i

$$\vdots \quad X_2 \rightrightarrows X_1 \underset{d}{\rightrightarrows} X_0$$

any morphism in Δ is a composite of d^i, s^i.

degen simplices $\bigcup_0^{n-1} s_i X_{n-1} = \bigcup_{\substack{\phi : [n] \to [k] \\ k < n,\ \phi \text{ surj}}} \phi^* X_k$

Example: define $\Delta^n = h_{[n]} = \Delta(\ , [n]) : \Delta^{op} \to \mathrm{Set}$

ie $\Delta^n \in \Delta^{op}\mathrm{Set}$ "n simplex"

Yoneda lemma $\Rightarrow$ $\Delta^{op}\mathrm{Set}(\Delta^n, X) = X_n$

explicitly, $(\Delta^n)_m = \Delta([m],[n])$

so $(\Delta^n)_0 = [n]$ $(\Delta^n)_n$ has a unique non-degenerate simplex, etc

write $\alpha : [m] \to [n]$ as $\alpha_0 \alpha_1 \dots \alpha_m$ where $\alpha_i = \alpha(i)$

	$m=2$	$m=1$	$m=0$
Δ^1	000 001 011 111	00 01 11	0 1

This is just the interval 

$\mathcal{C}$ small cat ("objects are a set")

$N(\mathcal{C}) \in \Delta^{op}Set$ "nerve of $\mathcal{C}$"

$(N\mathcal{C})_n = \mathrm{Funct}([n], \mathcal{C}) = \{ x_0 \xrightarrow{f_1} x_1 \to \dots \xrightarrow{f_n} x_n \}$ composable n arrows

degeneracies: insert identity map
face maps: composition

e.g. $\mathcal{C} = [n] = \{ 0 \to 1 \to 2 \to \dots \to n \}$ exercise $N\mathcal{C} - \Delta^n$

geometric realization

functor $\sigma_\bullet : \Delta \to Top$ $[n] \mapsto \sigma_n$ = convex hull of $e_0, \dots, e_n$ in $\mathbb{R}^{n+1}$

$\alpha : [n] \to [m]$ $\alpha_* : \sigma_n \to \sigma_m$ extend linearly

Gives functor $Sing : Top \to \Delta^{op}Set$ $Sing(X)_n = Top(\sigma_n, X)$

an adjoint functor. $|\cdot| : \Delta^{op}Set \to Top$ left adj to $Sing$

$$\coprod_{\phi:[n]\to[m]} X_m \times \sigma_n \rightrightarrows \coprod X_n \times \sigma_n \longrightarrow |X|$$

i.e., take a geometric n-simplex for each $x \in X_n$ & glue by boundary maps.

If we take Top to mean compactly generated weak Hausdorff spaces, then Top is Cartesian closed (has products and mapping spaces)

& then: (i) $|\Delta^n| = \sigma_n$

(ii) $|X \times Y| \longrightarrow |X| \times |Y|$ is a homeo [Eilenberg-Zilber Thm?]

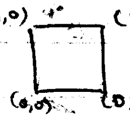
(iii) $|\cdot| : \Delta^{op}Set \rightleftarrows Top : Sing$ are adjoint

$Top(|X|, Y) \cong \Delta^{op}Set(X, Sing\, Y)$

Defn $(X \times Y)_n := X_n \times Y_n$, $\alpha^*_{X\times Y} = \alpha^*_X \times \alpha^*_Y$ $\alpha : [n] \to [m]$

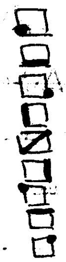
Product structure of simplicial objects

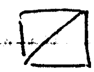
$(\Delta^1 \times \Delta^1)_0 =$ (0,0) (0,1) (1,0) (1,1)



$(\Delta^1 \times \Delta^1)_1 =$ 00, 00
00, 01
00, 11
01, 00
01, 01
01, 11
11, 00
11, 01
11, 11

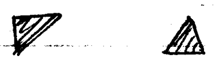
ac bd
↑↑
(ab, cd)





So we have 5 nondeg 1-simplices
4 deg.

$(\Delta^1 \times \Delta^1)_2$



(011, 001) (001, 011)

Exercise K a simplicial complex, order set of simplices, call it J. regard it as a cat $\mathcal{C}$. Then $|N\mathcal{C}|$ is the Barycentric subdiv of K. In particular, homeo to K. I.e. any simplicial complex (cw complex) is homotopic to $|N\mathcal{C}|$, some cat $\mathcal{C}$, or $|X|$, some sSet.

END OF REVIEW / PREAMBLE

§ $(\infty,1)$-categories, after Rezk

recap: $N : \underset{\sim}{\mathcal{C}at} \longrightarrow \Delta^{op}\mathrm{Set}$, $\mathcal{C} \rightsquigarrow N\mathcal{C}$

& if we want, we can pretend $N\mathcal{C}$ is a top space $|N\mathcal{C}|$, well-defined up to homotopy.

Properties of N: (i) natural iso's $N(\mathcal{C}\times\mathcal{D}) \xrightarrow{\sim} N\mathcal{C} \times N\mathcal{D}$

[(ii) $N(\mathcal{D}^{\mathcal{C}}) \xrightarrow{\sim} N(\mathcal{D})^{N(\mathcal{C})}$]

clear N embeds $\mathcal{C}at \hookrightarrow \Delta^{op}\mathrm{Set}$

i.e. $N\mathcal{C} \underset{\uparrow}{=} N\mathcal{C}' \Rightarrow \mathcal{C} = \mathcal{C}'$

equal, not isom

image of N also clear: $\{X \in \Delta^{op}\mathrm{Set} \mid X_n = X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1\}$

but if we want to recover Cat/equiv, not Cat/equality [equality = naturally iso]

Consider $\mathcal{C} \rightsquigarrow |N\mathcal{C}|$ [Think Vect vs. Δ]

Claim (i) $\mathcal{C} \underset{\text{equiv}}{\sim} \mathcal{C}' \Rightarrow |N\mathcal{C}| \underset{\text{homotopic}}{\sim} |N\mathcal{C}'|$

so homotopy type of $|N\mathcal{C}|$ is an invariant of $\mathcal{C}$/equiv

(ii) loses lots of information

e.g. (ii) $\mathcal{C} \not\simeq \mathcal{C}^{op}$ in general, but $|N\mathcal{C}| = |N\mathcal{C}^{op}|$ // you forgot the arrows when $|\cdot|$

to prove (i), & for more example,

Lem A natural transform. $\Theta : F \longrightarrow G : \mathcal{C} \longrightarrow \mathcal{D}$

induces a homotopy $|N\mathcal{C}| \times [0,1] \longrightarrow |N\mathcal{D}|$ between $|F|$ & $|G|$

Pf $[1] = \{0 \longrightarrow 1\}$ cat

Natural transform defines a functor $\Theta : \mathcal{C} \times [1] \longrightarrow \mathcal{D}$

hence as N, $|\cdot|$ preserve products

$$|N\mathcal{D}| \longleftarrow |N(\mathcal{C}\times[1])| \xrightarrow{\sim} |N\mathcal{C} \times N[1]| \xrightarrow{\sim} |N\mathcal{C}| \times |N[1]| \xrightarrow{\sim} |N\mathcal{C}| \times [0,1]$$

get homotopy. ∎

Cor $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ adjoint $\Rightarrow |N\mathcal{C}| \sim |N\mathcal{D}|$

Pf adjoint $\Rightarrow 1 \longrightarrow GF$, $FG \longrightarrow 1$ nat. transf. ∎

// fact that these unit/counit look familiar to 2-TFT is no coincidence!

Sub-eg. If $\mathcal{C} \sim \mathcal{C}'$, then $|N\mathcal{C}| \sim |N\mathcal{C}'|$

Cor: If $\mathcal{C}$ has an initial or a final obj, then $|N\mathcal{C}|$ is contractible

Pf Functor to the one object, one morphism cat $[0]$ has an adjoint.

E.g. If $\mathcal{C}$ additive cat, e.g. vect spaces, or chain complexes, ... $|N\mathcal{C}| \sim *$

Issue: If $X \in \mathrm{Top} \rightsquigarrow$ cat $\underset{\sim}{\pi_{\leq 1} X}$ in which all morphisms are invertible (groupoid)

obj: pts of X

mor: $x \longrightarrow y = \{$cts map $f : [0,1] \longrightarrow X$, $f(0) = x$, $f(1) = y\}$ / homotopy so $|N-|: \mathcal{C}at \to Top$ takes us to a place where all morphisms are invertible.

Notation: If G discrete gp, BG = cat with $ob\, BG = *$, $BG(*,*) = G$

$\cancel{4}$

$\mathcal{C}^{inv}$: $ob\, \mathcal{C}^{inv} = ob\, \mathcal{C}$
$\mathcal{C}^{inv}(x,y) = \{\varphi \in \mathcal{C}(x,y) \mid \varphi \text{ invertible}\}$

this is a groupoid "throw away all non-invertible" morphisms

$|N\mathcal{C}^{inv}|$

$FinSets^{inv} \sim \coprod_{n\geq 0} BS_n$ not contractible $\rightsquigarrow$ you get nice top. space.

4/4

// combinatorics of higher cat: Θ iterated wreath product of Δ

Lecture 3

01/26/12

$\mathcal{C} \in \underline{\mathcal{C}at} \rightsquigarrow \mathcal{C}^{inv}$ subcat: $\cdot$ same objects
$\mathcal{C}^{inv}(X,Y) = \{\varphi \in \mathcal{C}(X,Y) \mid \varphi^{-1} \text{ exists}\}$

$\mathcal{C} \to \mathcal{C}^{inv}$ functor, $\mathcal{C} \sim \mathcal{C}' \Rightarrow \mathcal{C}^{inv} \sim \mathcal{C}'^{inv}$ $\sim$ means equivalent

e.g.: $FinVect_k \sim StrVect_k$: $ob = \mathbb{N}$ $Mor(m,n) = n\times m$ matrices $(\cong Hom(k^m, k^n))$

$FinVect_k^{inv} \sim StrVect_k^{inv} = \coprod_{n\geq 0} B[GL_n(k)^{disc}]$ // disc = discrete

(cat with objects $n \in \mathbb{N}$ $Mor(n,n) = GL_n(k)$)

so $\mathcal{C} \rightsquigarrow N(\mathcal{C}^{inv})$ loses less info

Observation: if $\mathcal{C}$ is a connected groupoid
& $x \in ob\, \mathcal{C}$, $B(Aut\, x) = B\mathcal{C}(x,x) \hookrightarrow \mathcal{C}$ is an equiv of cat.

so $\mathcal{C}^{inv} \sim BAut(x)$

hence in general
$\mathcal{C}^{inv} \sim \coprod_{x \in ob\, \mathcal{C}/iso} BAut(x)$: choose a section $ob\, \mathcal{C}/iso \hookrightarrow ob\, \mathcal{C}$.

non-invertible morphisms?

Consider $Funct([1], \mathcal{C}) = \mathcal{C}^{[1]}$: Objects $= \coprod_{x,y \in ob\, \mathcal{C}} \mathcal{C}(x,y)$
($[1]$: $0 \to 1$)

morphisms
$$\begin{tikzcd} x \arrow[r, "\varphi"] \arrow[d, "f"'] & y \arrow[d, "f'"] \\ x' \arrow[r, "\varphi'"'] & y' \end{tikzcd}$$
st. diagram commutes

Apply $()^{inv}$, ie. require f, f' to be iso's,

Examples: (i) $\mathcal{C} = StrVect_k$,

$$ob\, \mathcal{C}^{[1]} = \coprod_{n,m} Hom(k^n, k^m)^{disc}$$

morphisms
$$\begin{tikzcd} k^n \arrow[r, "A"] \arrow[d, "X"'] & k^m \arrow[d, "Y"] \\ k^n \arrow[r, "B"'] & k^m \end{tikzcd}$$
$YA = BX$

If invertible, $(X,Y) \in GL_n \times GL_m$.

so $(\mathcal{C}^{[1]})^{inv} = \coprod_{n,m} Hom(k^n, k^m)^{disc} / (GL_n \times GL_m)^{disc}$

$(X,Y)\cdot A = YAX^{-1}$

So iso classes of objects are orbits of $G = GL_n \times GL_m$ on $E = Hom(k^n, k^m)$

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$= \{d \in \mathbb{N} \mid 0 \le d \le \min(m,n)\}$

$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \ddots \; 0 \end{pmatrix}$ $\quad A \;\; A_d$; $\;A \leftrightarrow \operatorname{rk} A$, $\;A_d \leftarrow d$

So $\operatorname{Aut}(A_d) = \{(X,Y) \in GL_n \times GL_m \mid YA_d = A_d X\}$

$= GL_d \times GL_{n-d} \times GL_{m-d} \rtimes k^{d(n-d)} \times k^{d(m-d)}$

So $(\mathrm{FinVect}^{[1]})^{\mathrm{inv}} \sim \coprod_{\substack{m,n,d \\ 0 \le d \le \min(m,n)}} B\operatorname{Aut}(A_d)$

(ii) $\mathcal{C} = \mathrm{FinSet}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} \\ X & \xrightarrow{f} & Y \end{array}$$

$X \overset{f}{\downarrow} Y$

clear that $\operatorname{Aut}(f) = (S_3 \times S_3) \rtimes S_2 \times S_2$

In general, iso type of $f : X \to Y \leftrightarrow (f_1, \dots, f_{\#X}) \in \mathbb{N}^{\#X}$ st. $\sum i f_i = \#X$

$f_i = \# Y_i$, $Y_i = \{y \in Y \mid \# f^{-1}(y) = i\}$

wreath prod

$\operatorname{Aut}(f) = \prod_k \left[\prod_{y \in Y_k} \operatorname{Sym} f^{-1}(y)\right] \rtimes \operatorname{Sym} Y_k \Big] = \prod_k S_k \wr S_{f_k}$

(iii) $\mathcal{C} = BG$, G discrete gp

$\operatorname{ob}(\mathrm{Func}([1], G)) = \{ * \xrightarrow{g} * \mid g \in G\} = G$

$\operatorname{Aut}(* \xrightarrow{g} *) = \left\{ \begin{array}{ccc} \cdot & \xrightarrow{g} & \cdot \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \cdot & \xrightarrow{g} & \cdot \end{array} \;\middle|\; \begin{array}{l} h'g = gh \\ \text{ie. } h' = ghg^{-1} \end{array} \right\}$

$= G$

so $(BG^{[1]})^{\mathrm{inv}} \sim BG$ so no new info.

Should have mentioned earlier: $\pi_1 |N(BG)| = G$, so lose nothing.

(iv) $\mathcal{C} = [n]$, $\mathrm{Funct}([1],[n])^{\mathrm{inv}} =$ discrete cat $\Delta([1],[n]) = [n]_1$

(with no non-id morphisms)

---

now, $d^0, d^1 : [0] \to [1]$

$\rightsquigarrow \mathrm{Funct}([1], \mathcal{C}) \rightrightarrows \mathrm{Funct}([0], \mathcal{C}) = \mathcal{C}$

so apply $N[(\cdot)^{\mathrm{inv}}]$

// $Sp := \Delta^{op}\mathrm{Set}$ but you are free to think about as topological spaces, eg. via geometric realization

We may as well consider $\mathrm{Funct}([n], \mathcal{C})^{\mathrm{inv}} = (\mathcal{C}^{[n]})^{\mathrm{inv}}$

define $\mathcal{N}\mathcal{C}_n = N((\mathcal{C}^{[n]})^{\mathrm{inv}}) \in Sp$

~~so $\mathcal{N}\mathcal{C} \in Sp = \Delta^{op}\mathrm{Set}$~~

(if want, let's also define $|\mathcal{N}|\mathcal{C}_n = |\mathcal{N}\mathcal{C}_n| \in \mathrm{Top}$

i.e. $\mathcal{C}$ cat $\rightsquigarrow \mathcal{N}\mathcal{C}_\bullet \in \Delta^{op}Sp = \Delta^{op}\Delta^{op}\mathrm{Set}$

$\cdots \mathcal{N}\mathcal{C}_2 \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{N}\mathcal{C}_1 \rightrightarrows \mathcal{N}\mathcal{C}_0$

This is called simplicial nerve.

Exercise: If $\mathcal{C} = BG$, G discrete gp

$\mathcal{N}BG = \left(\cdots \substack{\to\\\to\\\to} N(BG) \rightrightarrows N(BG) \right)$ is the constant simplicial group.

(if $\mathcal{C}$ is a cat, $\mathcal{C} \longrightarrow \Delta^{op}\mathcal{C}$ "constant simplicial object"

$X \longmapsto \left([n] \mapsto X, \ \alpha: [n] \to [m], \ \alpha^* = \mathrm{Id}: X \to X \right)$)

Exercise (iv) if $\mathcal{C}$ is a cat s.t. only iso's are identity morphism.

(example $\mathcal{C}$ cat attached to poset)

e.g. $\mathcal{C} = [n]$ get nerve of $\mathcal{C}$, embedded "horizontally" in $\Delta^{op}\Delta^{op}Set$

$(\Delta \to \mathcal{C} \hookrightarrow \Delta^{op}\mathcal{C}, \ \mathcal{C} = Set)$ not "vertically" [think bi-simplicial set]

show $\mathcal{N}[k]_n$ = set $\Delta([n],[k]) \hookrightarrow \Delta^{op}Set$

thought of as a const simplicial set

which is the element of $\Delta^{op}Sp$ which "represents"

$X \mapsto X_k : \Delta^{op}Sp \longrightarrow Sp$

call that $F(k) : ([n] \to \Delta(n,k))$, so $F(k) \in \Delta^{op}Sp$

$\Delta^k \in \Delta^{op}Set$

☆ **Exercise** (i), (ii) FinVect, FinSet

[For FinVect:

$k^{n_1} \to k^{n_2} \cdots k^{n_d}$

$E = \prod Hom(k^{n_i}, k^{n_{i+1}})$ $G = \prod GL_{n_i}$

$\mathbb{C}^{n_1} \to \mathbb{C}^{n_2} \cdots \mathbb{C}^{n_{d-1}} \to \mathbb{C}^{n_d}$

$\alpha_i = \alpha_{i,i+1}$

// Quiver

Gabriel's thm

E/G Positive roots in sl_{d+1}

$\sum_{\alpha \in \Phi^+} n_\alpha \alpha$ $n_\alpha \geq 0$]

// Out pops $U(\mathfrak{n})$ $\mathfrak{n}$ for sl_{d+1} [??]

$\mathcal{C}$ cat, $\mathcal{N}\mathcal{C} \in \Delta^{op}(Sp)$

? note $\mathcal{N}\mathcal{C}_n \xrightarrow{\sim} \mathcal{N}\mathcal{C}_1 \times_{\mathcal{N}\mathcal{C}_0} \mathcal{N}\mathcal{C}_1 \times_{\mathcal{N}\mathcal{C}_0} \cdots \times_{\mathcal{N}\mathcal{C}_0} \mathcal{N}\mathcal{C}_1$ canonically

Even better: $\boxed{\mathcal{N}\mathcal{C}_1 \to \mathcal{N}\mathcal{C}_0 \times \mathcal{N}\mathcal{C}_0 \text{ is a fibration}}$

[fiber over (x,y) is $\mathcal{C}(x,y)$

& so depends only on class of x,y in $\pi_0(\mathcal{N}\mathcal{C}_0) \times \pi_0(\mathcal{N}\mathcal{C}_0)$]

up to iso

because of this, the fibre product is a homotopy fiber product

EXPLANATION:

$$\begin{array}{ccc} \cdot & \longrightarrow & X' \\ \downarrow & \square & \downarrow f' \\ X & \xrightarrow{f} & Z \end{array}$$

$X \times_Z X' = \{(x,x') \mid f(x) = f'(x')\}$

pullback of top spaces has the property that it depends on more than the homotopy type of X, X', Z.

e.g.

$$\begin{array}{ccc} & & \{1\} \\ & & \downarrow \\ \{0\} & \longrightarrow & [0 \to 1] \end{array}$$

pullback is $\emptyset$

pullback is $[0,1]$

but everything is homotopic to pt.

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To fix this, the "homotopy pullback"

$$X \overset{h}{\underset{Z}{\times}} X' = X \underset{Z}{\times} \mathcal{P}Z \underset{Z}{\times} X',$$

where $\mathcal{P}Z = \{\alpha : [0,1] \to Z\}$

$$\begin{tikzcd} & \mathcal{P}Z \arrow[dl] \arrow[dr] & \\ Z & & Z \end{tikzcd} \qquad \begin{tikzcd} & \alpha \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \alpha(0) & & \alpha(1) \end{tikzcd}$$

notice $X \underset{Z}{\times} X' \longrightarrow X \overset{h}{\underset{Z}{\times}} X'$ inclusion

Not always a homotopy equiv.

has following properties:

(i) if
$$\begin{tikzcd} X \arrow[r] & Z & X' \arrow[l] \\ Y \arrow[r] \arrow[u] & W \arrow[u] & Y' \arrow[l] \arrow[u] \end{tikzcd}$$
st. vertical maps are weak homotopy equiv.

then $Y \overset{h}{\underset{W}{\times}} Y' \to X \overset{h}{\underset{Z}{\times}} X'$ is a weak homotopy equiv.

(ii) if $X \to Z$ is a fibration, then $X \underset{Z}{\times} X' \to X \overset{h}{\underset{Z}{\times}} X'$ is a homotopy equiv.

Look in Goerss paper for model cat struct. on Sp.

Dwyer–Spalinski — another expository paper on model cats (w/ proofs)

$\mathcal{NC}_\bullet$ enable us to do computation on cats in homotopy invariant way.

def $X_\bullet \in \Delta^{op}Sp$ satisfies the Segal condition if the map

$$X_n \to X_1 \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$$

is a weak equiv in Sp, $\forall n$.

(e.g. $\mathcal{NC}_\bullet$)

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Lecture 4

01/31/12

? Homotopy fibered prod in Sp?

def $X \in \Delta^{op}Sp$ satisfies the Segal condition if $X_n \to X_1 \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$ is a weak equiv in Sp, $\forall n$

call such a "Segal space"

homotopy fiber: not defined for Sp, but just think of Top. after geom realization

Such a space has a weak notion of composition:

get a "category" whose objects are pts in X_0

If $x, y \in X_0$, put $\mathrm{map}_X(x,y) = $ homotopy fiber $\left(\begin{tikzcd} & \{(x,y)\} \arrow[d] \\ X_1 \arrow[r] & X_0 \times X_0 \end{tikzcd} \right)$

i.e. if $X_1 \xrightarrow[(d_1, d_0)]{} X_0 \times X_0$ is a fibration, then $\mathrm{map}_X(x,y) = (d_1, d_0)^{-1}(x,y)$

Whether or not its a fibration, homotopy ~~fiber~~ type of $\mathrm{map}_X(x,y)$ depends only on the component of $\pi_0(X_0 \times X_0)$ containing (x,y).

def If X is a Segal space, $f, g \in \mathrm{map}_X(x,y)$

say $f \sim g$ "f homotopic to g" if f, g lie in the same component of $\mathrm{map}_X(x,y)$.

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"composition"

If $\alpha : [n] \to [m]$,

$\alpha(i) = \alpha_i$

Write $\delta^{\alpha_0 \cdots \alpha_n}$

for α

$$\begin{array}{ccc} X_2 & \xrightarrow{X(\delta^{02})} & X_1 \\ {\scriptstyle S}\downarrow X(\delta^{01}, \delta^{12}) & & \\ X_1 \times_{X_0} X_1 & & \end{array}$$

If $f \in map_X(x,y)$, $g \in map_X(y,z)$

choose a point $\gamma \in X(\delta^{01}, \delta^{12})^{-1}(f,g)$

and take $X(\delta^{02})(\gamma)$ to be $g \circ f$, the composition.

As fibers of $X(\delta^{01}, \delta^{12})$ are contractible, any other choice of γ

gives a map in the same component of $map_X(x,z)$, ie. a homotopic map.

exercise (i) If $g \circ f$ denotes any such choice, show $(g \circ f) \circ h \sim g \circ (f \circ h)$

by showing you can actually make these equal. by choosing a lift

of $g \circ f \circ h$ to X_3.

(ii) Show $f \circ Id_x \sim f$, $Id_y \circ f \sim f$ by using degeneracy map to lift to an equality

cor define $ho(X)$, X a Segal space, to be the honest category

$ob\, ho(X) = X_0$, $ho(X)(a,b) = \pi_0\, map_X(a,b)$

Then exercise shows you $ho(X)$ is a category.

? Moreover, $ho(\mathcal{N}\mathcal{C}) \cong \mathcal{C}$ canonically

so $\mathcal{N} : \mathcal{C}at \to \Delta^{op} Sp$ is a full embedding. (ie. fully faithful)

def $f : X \to Y \in \Delta^{op} Sp$ is a "level-wise weak equiv" if

$\forall n$, $f_n : X_n \to Y_n$ is a weak equiv in Sp.

Prop Let $\mathcal{C}, \mathcal{D}$ be categories. Then

look up defn of mapping obj in Sp

(i) $\mathcal{N}(\mathcal{C} \times \mathcal{D}) \cong \mathcal{N}\mathcal{C} \times \mathcal{N}\mathcal{D}$, $\mathcal{N}(\mathcal{C}^{\mathcal{D}}) \cong (\mathcal{N}\mathcal{C})^{\mathcal{N}\mathcal{D}}$ canonical isos

(ii) $\mathcal{N}\mathcal{C}$ is a Reedy fibrant simplicial space

(iii) $F : \mathcal{C} \to \mathcal{D}$ is an equiv of cats $\Leftrightarrow \mathcal{N}F : \mathcal{N}\mathcal{C} \to \mathcal{N}\mathcal{D}$ is a level wise weak equiv.

Pf. (i) Products: Clear.

Recall/Observe m-simplices of $\left((\mathcal{N}\mathcal{C})_n = N((\mathcal{C}^{[n]})^{inv}) \right)$

$= Funct([n] \times I(m), \mathcal{C})$

where $I(m)$ = cat with $m+1$ distinct objects, & a unique iso between any two objects.

We must show that following are iso

m-simplices of $\mathcal{N}(\mathcal{C}^{\mathcal{D}})_n = Funct([n] \times I(m), \mathcal{C}^{\mathcal{D}}) = (**)$

- - - - - - - - $(\mathcal{N}(\mathcal{C})^{\mathcal{N}\mathcal{D}})_n = \text{Maps}(F(n) \times \Delta^m, \mathcal{N}\mathcal{C}^{\mathcal{N}\mathcal{D}}) = (*)$

Recall that $F(n)$ represents $X \to X_n : \Delta^{op}Sp \to Sp$ (and was Δ^n horizontally)

(better notation: if $X \in \Delta^{op}$ Set, write $X^+ \in \Delta^{op} Sp$ $\mathcal{N}[n]$

for $(X^+)_n$ = const simplicial set X_n)

But $\mathcal{N}([n]\times\mathcal{C}) = \mathcal{N}[n]\times\mathcal{N}\mathcal{C}$, by product

• (1) $= F(n)\times\mathcal{N}\mathcal{C}$

Check! • (2) $\mathcal{N}(\mathcal{C}^{I(n)}) = (\mathcal{N}\mathcal{C})^{\Delta^n}$ as $\mathrm{isos}(\mathcal{D}^{I(n)}) = \mathrm{isos}(\mathcal{D})^{I(n)} = \mathrm{iso}(\mathcal{D})^{[n]}$

Now take $\mathcal{D} = \mathcal{C}^{[m]}$

and use $\mathcal{N}$ fully faithful

So $(*) = \mathrm{Maps}(\mathcal{N}([n]\times I(m)), \mathcal{N}(\mathcal{C}^{\mathcal{D}}))$ by A LOT of adjunctions in the right order.

but we just showed $\mathcal{C}\sim\mathcal{N}\mathcal{C}$ is a full embedding of cats, so

$(*)$ & $(**)$ canonically iso

(ii) is a technical statement, that certain maps $(\mathcal{N}\mathcal{C})_n \longrightarrow M_n(\mathcal{N}\mathcal{C})$ are fibrations

here $M_nX = \lim_{\substack{\varphi:[k]\to[n] \\ k<n,\ \varphi \text{ injection}}} X_k$

means

$n=0$: $(\mathcal{N}\mathcal{C})_0 = N(\mathcal{C}^{\mathrm{iso}})$ is a groupoid, hence a Kan complex.

$n=1$: $\mathcal{N}\mathcal{C}_1 \longrightarrow \mathcal{N}\mathcal{C}_0\times\mathcal{N}\mathcal{C}_0$ is a fibration.

$n=2$: M_2 is an inclusion of path components, so a fibration

$n\geq 3$: M_n is an iso, so a fibration. ∎

(iii) as $\mathcal{N}(\mathcal{C}^{I(1)}) = (\mathcal{N}\mathcal{C})^{\Delta^1}$

then (just as with N) an equivalence of cats induces a simplicial homotopy of simplicial spaces, & so a levelwise weak equiv.

Conversely, if $\mathcal{N}F: \mathcal{N}\mathcal{C}\to\mathcal{N}\mathcal{D}$ is a levelwise weak equiv,

then because $\mathcal{N}\mathcal{C}, \mathcal{N}\mathcal{D}$ are Reedy Fibrant, $\mathcal{N}F$ is actually a simplicial homotopy equiv. // this is the homotopical version of: quasi-isom between // injective chain complexes is a map chain homotopic to 1

Moreover, the homotopy inverse is a 0-simplex of $\mathcal{N}(\mathcal{C}^{\mathcal{D}})_0 = \mathrm{Maps}(\mathcal{N}\mathcal{D},\mathcal{N}\mathcal{C}) = \mathrm{Funct}(\mathcal{D},\mathcal{C})$

& the simplicial homotopies are 1-simplices of $\mathcal{N}(\mathcal{D}^{\mathcal{C}})_0$, $\mathcal{N}(\mathcal{C}^{\mathcal{D}})_0$

& hence by what we've done correspond precisely to $G:\mathcal{D}\to\mathcal{C}$ functor,

& natural iso's $FG\longrightarrow 1$, $GF\longrightarrow 1$ as needed. ∎

Example a discrete simplicial space $X\in\Delta^{op}Sp$ is one with $X_n\in Set \hookrightarrow \Delta^{op}Set = Sp$ $\forall n$

(i.e. one of the form Y^+, $Y\in\Delta^{op}Set$)

Exercises (i) Show a discrete simplicial space is always Reedy fibrant.

(ii) if X is a discrete $\Delta^{op}Sp$, then X satisfies the Segal condition iff $X_n \longrightarrow X_1\times_{X_0}\cdots\times_{X_0}X_1$ is an iso.

Hence a discrete simplicial space X satisfies Segal condition $\iff X = (N\mathcal{C})^+$

Coincidentally, this shows $F(n) = (\Delta^n)^+$ satisfies Segal condition.

So both $(N\mathcal{C})^+$ & $\mathcal{N}\mathcal{C}$ Reedy fibrant Segal Spaces, so we're missing one more condition "completeness condition"

X a Segal space, Reedy fibrant

- if $\alpha \in X_1$ is a homotopy equiv from $d^0\alpha \to d^1\alpha$
  (i.e. image of α in $ho(X)(d^0\alpha, d^1\alpha)$ invertible)
  & β is in the same component as α, then β also does.

? Let $(X_1)_{\text{ho equiv}}$ = components of X_1 st. $[\alpha]$ invertible $\forall \alpha$ in the component.

Notice that degeneracy map $X_0 \xrightarrow{s_0} X_1$ factors through $(X_1)_{\text{hoequiv}}$
as $(s_0 x) = \mathrm{Id}_x \in ho(X)(x,x)$

def A Segal space is "complete" if $X_0 \longrightarrow (X_1)_{\text{hoequiv}}$ is a weak equiv in Sp.
i.e. if X_0 is already the ~~mo~~duli space of all invertible maps in $ho(X)$,

call X st.
- Reedy fibrant
- Segal
- complete

"complete Segal spaces" CSS

Rezk's Thesis

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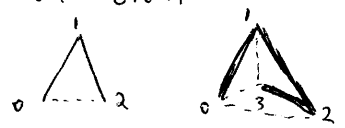
Lecture 5 $\mathcal{C} \rightsquigarrow N\mathcal{C} \in \Delta^{op}Sp$ is CSS. 02/02/12

Segal condition, rephrased:

let $G(k) \subseteq F(k) = (\Delta^k)^+ \in \Delta^{op}Sp$ be the path "from 0 to k" in the k-simplex



i.e. $G(k) = \underset{(\text{homotopy})}{\text{colim}} \left(\begin{tikzcd}[column sep=small] & F(0) \arrow[dl, "Fd_1"'] \arrow[dr, "Fd_0"] & & F(0) \arrow[dl, "Fd_1"'] \arrow[dr] & & & F(0) \arrow[dl, "Fd_0"'] \arrow[d, "Fd_0"] \\ F(1) & & F(1) & & \cdots & F(1) & F(1) \end{tikzcd} \right)$

and $G(k) \longrightarrow F(k)$ via $(F(\delta^{01}), F(\delta^{12}), \ldots, F(\delta^{k-1,k}))$

(as a simplicial space, its $\bigcup_{0}^{k-1} \delta^{i,i+1} F(1) \subseteq F(k)$)

Then $\mathrm{Maps}_{\Delta^{op}Sp}(G(k), X) = \lim \left(X_1 \searrow X_0 \swarrow X_1 \searrow X_0 \swarrow X_1 \searrow \ldots \searrow X_0 \swarrow X_1 \right)$
as $\mathrm{Maps}(\cdot, X)$ takes homotopy colimit to limit

& $\mathrm{Maps}_{\Delta^{op}Sp}(F(k), X) = X_k$

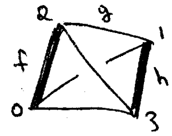
So Segal Condition is precisely map $\boxed{G(k) \longrightarrow F(k)}$
induces a weak equiv $\mathrm{Map}_{\Delta^{op}Sp}(F(k), X) \longrightarrow \mathrm{Maps}_{\Delta^{op}Sp}(G(k), X)$.

Similarly, $\exists Z \in \Delta^{op}Sp$, and maps
$Z \longrightarrow F(0)$, $F(1) \longrightarrow Z$ st.

Prop If X satisfies the Segal condition, then $\mathrm{Maps}_{\Delta^{op}Sp}(Z, X) \to \mathrm{Maps}_{\Delta^{op}Sp}(F(1), X) = X_1$
factors through $(X_1)_{\text{hoequiv}}$ & is in fact a weak equiv.
with $(X_1)_{\text{hoequiv}}$

hence, if X is Segal: the map $Z \longrightarrow F(0)$ induces a weak equiv
$\mathrm{Maps}(F(0), X) \longrightarrow \mathrm{Maps}(Z, X)$
$\Leftrightarrow X$ is complete.

What is Z? (since you asked...)



Z = 3-simplex/$\sim$

$$= \operatorname{colimit}\left(\begin{tikzcd} & F(1) \amalg F(1) \arrow[dl, "\delta^{02}, \delta^{13}"'] \arrow[dr, "\delta^{00} \amalg \delta^{00}"] & \\ F(3) & & F(0) \amalg F(0) \end{tikzcd} \right)$$

(Show g is invertible up to homotopy iff $\exists$ $\alpha : gf \sim 1_x$
$\beta : hg \sim 1_y$
and set of (α, β, g, h) weakly contractible.)

// All 3-conditions of CSS can be rephrased as saying
// finite # of maps when you apply $\mathrm{Maps}(-, X)$ become weak equiv

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Localization of Model Cats

Let M be a model cat, eg. $\Delta^{op}Sp$, Top, $Ch(R)$, ...
S be a set of morphisms in M. [want to "invert morphisms in S"]

def $X \in M$ is "S-local" if $\forall$ $\alpha : s \to s'$ in S,
$\alpha^* : map(s', X) \to map(s, X)$ is a weak equiv. in M.

// M model cat enriched over another model cat (so weak equiv of maps makes sense)
// or M cartesian closed

example $M = \mathbb{C}[x]$-mod.
$S = \{\text{mult by } x,\ x : \mathbb{C}[x] \to \mathbb{C}[x]\}$

So $N \in \mathbb{C}[x]$-mod is S-local if $x : N \to N$ is a "weak equiv." (isom in Mod, q-i in Ch)

$\Leftrightarrow$ N is a $\mathbb{C}[x, x^{-1}]$-mod

so ~~S-local objects means~~ morphisms of S are invertible on S-local objects
and if M_S = full subcat of M consisting of S-local objects
is a good model for "$M[S^{-1}]$"

Put $\bar{S} = \{\alpha : s \to s' \text{ in } M \mid \alpha^* : maps(s', X) \to maps(s, X) \text{ is a w.e. if } X \text{ is } S\text{-local}\}$

so S-local objects X $\leftrightarrow$ $\bar{S}$-local objects X

so "$M[S^{-1}] = M[\bar{S}^{-1}]$"

thm: Given (M, S) satisfying some conditions. [left proper, tractable]
there is a new model cat structure on M st.
- weak equivs are the S-local maps
- cofibs are as before

Moreover, fibrant objects are S-local objects which are already fibrant in M.
Furthermore, if M is Cartesian, then this new model cat is cartesian iff
$\alpha : s \to s' \in S \Rightarrow \alpha \times \mathbf{1}_x \in \bar{S}$, $\forall x \in ob\, M$.

$x \xrightarrow{\sim} x^f \twoheadrightarrow *$ so in particular, the thm is saying
if $N \in M$, $\exists$ $N \to N^f$ a weak equiv, with N^f S-local.
(i.e. that M_S is big enough!)

example: $\mathbb{C}[x]$-mod, $\quad N \longrightarrow N \otimes_{\mathbb{C}[x]} \mathbb{C}[x,x^{-1}] = \varinjlim_{x} N = \varinjlim (N \xrightarrow{x} N \xrightarrow{x} \cdots)$

// So idea is to take limits over and over; careful: must avoid set-theoretic difficulties!

// One way to avoid: Groth. universes

// Curr. best way: Lurie's HTT

After construction, you have to show indep of choice of resolution, etc.

We won't go into proof...

Cor let $M = \Delta^{op}Sp$, $S = \{G(k) \to F(k),\ F(0) \to *\}$

we get there is a simplicial closed model cat str on $\Delta^{op}Sp$ s.t.

- fibrant objects are the CSS,
- cofibs are the mono's,
- the w.e. are the maps $f: X \to Y$ st. $\mathrm{Map}_{\Delta^{op}Sp}(f, W): \mathrm{Map}(Y,W) \to \mathrm{Map}(X,W)$ is a w.e. for every CSS W.

Moreover, a levelwise weak equiv between any X, Y is a CSS-weak equiv.

& conversely, if X & Y are CSS,

then a CSS weak equiv is just a levelwise weak equiv.

thm (Rezk) Moreover, CSS is Cartesian closed.

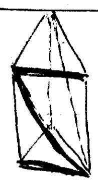
$f: X \to Y$ Segal spaces Dwyer–Kan equiv if $\mathrm{ho}X \to \mathrm{ho}Y$ equiv cat, $\mathrm{Map}(x,x') \to \mathrm{Map}(f(x), f(x'))$ w.e.

thm Dwyer-Kan equiv of CSS is levelwise w.e.

// inverting Dwyer–Kan equiv in Segal spaces gets CSS.

Thm basically says you can cut $G(n) \times F(m) \hookrightarrow F(n) \times F(m)$ into $G(k) \hookrightarrow F(k)$ pieces

e.g. $G(2) \times F(1)$



throw away bolded pieces

How to get CSS?

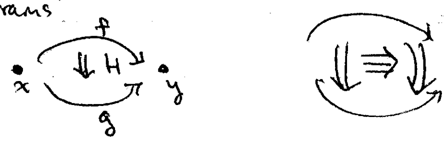
$(\mathcal{C}, \mathcal{W}) \rightsquigarrow (\mathcal{W}_{\mathcal{W}}\mathcal{C})_n$ arrows have to be in $\mathcal{W}$

some arrows wide subcat.

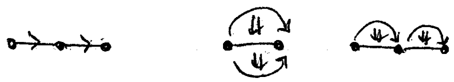
?? // [such pairs have model cat struct?]

n-Categories "globular diagrams"

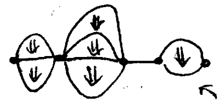
obj $\quad x \xrightarrow{f} y$ morphism



Compose:



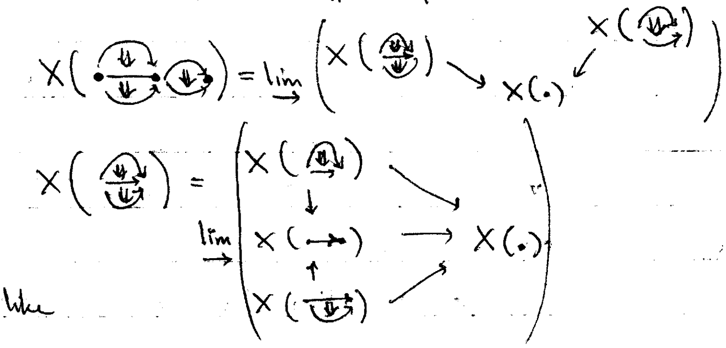
two ways

Strictness vs. non-strictness:  different ways to cut up."

roughly Θ_n = strict cat of such "pasting diagrams"
(play role of Δ)

Vague defn: Θ_n-space is a functor $X : \Theta_n^{op} \to Sp$

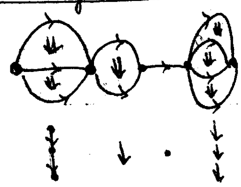
st. : Segal condition



Next time: what Θ_n looks like combinatorially.

Lecture 6

02/07/12

 $\in \Theta\Delta = \Theta_2$

$[4]([2],[1],[0],[3])$

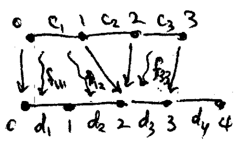
$\mathcal{C}$ small cat, "wreath product" $\Delta \wr \mathcal{C}$ denote $\Theta\mathcal{C}$
(Clemens Berger)

Cat : ob $\Theta\mathcal{C}$ tuples $[m](c_1, \dots, c_m)$ $\quad [m] \in \text{ob}\,\Delta$, ie $m \geq 0$
$c_1, \dots, c_m \in \text{ob}\,\mathcal{C}$

depict this $\bullet \xrightarrow{c_1} \bullet \xrightarrow{c_2} \cdots \xrightarrow{c_m} \bullet$ $\quad$ (write $[0] \in \Theta\mathcal{C}$)

morphisms $[m](c_1, \dots, c_m) \longrightarrow [n](d_1, \dots, d_n)$

e.g. $[3](c_1, c_2, c_3) \longrightarrow [4](d_1, d_2, d_3, d_4)$



i.e. a tuple $(\delta, (f_{ij}))$
where $\delta : [m] \to [n]$ morph in Δ

$\forall i,j \quad 1 \leq i \leq m, \ 1 \leq j \leq n$
s.t. $\delta(i-1) < j \leq \delta(i)$, a morphism
$f_{ij} : c_i \to d_j$ in $\mathcal{C}$

i.e. $$\mathrm{Mor}(\ ,\ ) = \coprod_{\delta : [m] \to [n]} \prod_{1}^{m} \prod_{j=\delta(i-1)+1}^{\delta(i)} \mathcal{C}(c_i, d_j)$$

- Composition is obvious [built out of comp in Δ & $\mathcal{C}$].

// This is literally the wreath product.

E.g. If $\mathcal{C} = * = [0]$ the terminal cat (one obj, one morph)
$\Theta(*) = \Delta$

def $\Theta_n = \Theta(\Theta_{n-1})$ and $\Theta_0 = *$

(so $\Theta_1 = \Delta$) (Θ_n are globular diagrams)

[Wreath products of categories over Segal category: associativity, etc]

Think of $\Theta\mathcal{C}$ as full subcat of $\mathcal{C}$-Cat (cat's enriched in $\mathcal{C}$)

<u>almost</u>

If $\mathcal{C}$ is Cartesian closed, with an initial object ϕ,

<u>exercise</u> (i) $\forall v \in ob\,\mathcal{C}$, $v \times \phi$ is initial

(ii) $\forall v \in ob\,\mathcal{C}$, $\mathcal{C}(v, \phi)$ is empty if v is not initial.

define $\tau : \Theta\mathcal{C} \to \mathcal{C}\text{-Cat}$

$[m](c_1, \dots, c_m) \mapsto$ free $\mathcal{C}$cat on graph $0 \xrightarrow{c_1} 1 \longrightarrow \cdots \xrightarrow{c_m} m$

- objects of this cat: $0, 1, \dots, m$

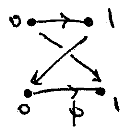
$$\mathrm{mor}(a,b) = \begin{cases} \phi & \text{if } b < a \\ \{1\} & a = b \\ c_{a+1} \times \dots \times c_b & \text{if } a < b \end{cases}$$

with obvious composition.

<u>exercise</u> If $\mathcal{V} \subseteq \mathcal{C}$ is a full subcat which does <u>not</u> contain any initial object, then $\tau : \Theta\mathcal{V} \to \mathcal{C}\text{-Cat}$ <u>is</u> fully faithful

<u>example</u> consider $\tau([1]\phi) = \tau(0 \xrightarrow{\phi} 1)$

objects $0, 1$



perfectly sensible morph in $\mathcal{C}$-Cat which sends $\phi : 0 \to 1$ to $\phi : 1 \to 0$

$\tau([1](A)) = \tau(0 \xrightarrow{A} 1)$ if A is not initial

$\mathrm{mor}(1,0) = \phi$, so no morphism $1 \to 0$, $0 \not\to 1$

as $\mathrm{mor}(0,1) = A$, & $\mathcal{C}(A, \phi)$ empty if A initial.

<u>ie. δ order</u> preserving automatically, & then exercise obvious

hence can regard

Θ_n as a full subcat of Strict-n-Cat

by $\tau_n : \Theta_n = \Theta(\Theta_{n-1}) \longrightarrow$ (Strict-$(n-1)$-Cat)-Cat

$\tau_n = \tau \circ \Theta(\tau_{n-1})$; (Strict-$(n-1)$-Cat)-Cat = Strict-n-Cat

$$sPSh(\mathcal{C}) = \text{Funct}(\mathcal{C}^{op}, Sp) = \text{Funct}(\mathcal{C}^{op}, \Delta^{op}\text{Set})$$

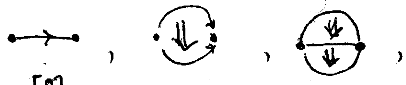
"simplicial presheaves on $\mathcal{C}$"

Idea: $\mathcal{C}$ cat, $sPSh(\Theta\mathcal{C})$ is a kind of weak "higher category"

if $X \in sPSh(\Theta\mathcal{C})$, "objects of X" are $X([0])$

"morphisms of X" to every $c \in ob\,\mathcal{C}$, a morphism space $X([1](c))$ labelled by c.

e.g. if $\mathcal{C} = \Delta$
"morphisms labelled by $n \in \mathbb{N}$"



$[0]$, $[1]$, $[2]$, ...

if $\mathcal{C} = \Theta\Delta$, morphisms labelled by 2-dim pasting diagram $[n]([a_1], \dots, [a_n])$

---

and then to $\bullet \xrightarrow{c_1} \bullet \xrightarrow{c_2} \bullet \rightsquigarrow X(\bullet \xrightarrow{c_1} \bullet \xrightarrow{c_2} \bullet) = X([2](c_1, c_2))$

space of composed morphisms, & so on.

i.e. $X \in sPSh(\Theta\mathcal{C})$ is a (weak) $sPSh(\mathcal{C})$-enriched cat

- define a localization of this (Segal + completeness conditions).

expect a map

$$sPSh(\mathcal{C})\text{-Cat} \longrightarrow sPSh(\Theta\mathcal{C})_{loc}$$

& model cat struct on which is a Quillen equiv.

e.g. If $\mathcal{C} = *$, $sPSh(*) = Sp$, $\Theta\mathcal{C} = \Delta$.

so expect a map $\underset{\text{"simplicial cats"}}{Sp\text{-Cat}} \longrightarrow sPsh(\Delta)_{loc} = CSS$ (thm: Rezk, J. Bergner)

when $\mathcal{C} = \Delta$, expect: $CSS\text{-Cat} \xrightarrow[??]{} sPSh(\Theta_2)_{loc}$

[$sPSh(\Theta_n)_{loc}$ will be our (∞, n)-cat]
fibrant objects in

If $\mathcal{C}$ cat, S set of morphisms in $sPSh(\mathcal{C})$.

Give $sPSh(\mathcal{C})$ injective model cat str (weak equiv, cofibrations are defined levelwise)

Properties: • cartesian closed, • every object is cofibrant
• discrete objects are fibrant

$$\left(\mathcal{C} \hookrightarrow \mathrm{Funct}(\mathcal{C}^{op}, Set) \hookrightarrow \mathrm{Funct}(\mathcal{C}^{op}, Sp)\right)$$
$$c \mapsto \left(F_{sPSh}^{\,c} : d \mapsto \mathcal{C}(d,c)\right)$$

In cases we care about, fibrations are Reedy.

$\rightsquigarrow$ new model cat $sPSh(\mathcal{C})_S^{inj}$ localized one

$(\mathcal{C}, S)$ " presentation of this model cat

If $(\mathcal{C}, S)$ presentation,

$\rightsquigarrow (\Theta\mathcal{C}, S_\Theta)$ new presentation

where $S_\Theta = \underline{Se^{\mathcal{C}}} \amalg \underline{V([1])(S)} \amalg \underline{Cpt_{\mathcal{C}}}$

(1) $Se^{\mathcal{C}}$ "Segal condition"

$$se^{(c_1, \dots, c_m)} = (F\delta^0, \dots, F\delta^{m-1,m}) : G[m](c_1, \dots, c_m) \longrightarrow F(m)(c_1, \dots, c_m)$$

$$G[m](c_1,\dots,c_m) = \mathrm{colim}\left(\begin{array}{ccccccc} & F(0) & & & F(0) & & \\ \swarrow F\delta^1 & & \searrow F\delta^0 & \cdots & \swarrow F\delta^1 & & \searrow F\delta^0 \\ F(1)(c_1) & & & & & & F(1)(c_m) \end{array} \right)$$

ie. just as before,

$X \in sPSh(\Theta\mathcal{C})$ which is inj fibrant is $Se^{\mathcal{C}}$ fibrant if

$$X([m](c_1, \ldots, c_m)) \xrightarrow[\text{w.e.}]{\sim} \lim \left(\begin{tikzcd}[column sep=tiny] X[1](c_1) \arrow[dr] & & X[1](c_2) \arrow[dl] \arrow[dr] & & X[1](c_m) \arrow[dl] \\ & X(0) & & \cdots & \end{tikzcd} \right)$$

(ii) Suspension morphism:

$V[1] : \mathcal{C} \to \Theta\mathcal{C}$

// by induction we have completeness for vertical, just need horizontal

(iii) completeness condition "at bottom level"

define an underlying simplicial space of $X : \Theta\mathcal{C}^{op} \to Sp$

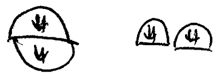
which by (i) will be a ~~CSS~~ Segal space

& we require it to be a complete "

thm (Rezk) (i) $(\Theta\mathcal{C}, Se_{\mathcal{C}})$ Cartesian

(ii) $(\Theta\mathcal{C}, Se_{\mathcal{C}} \amalg Cpt_{\mathcal{C}})$ & $(\Theta\mathcal{C}, S_{\Theta})$ are Cartesian (if $(\mathcal{C}, S)$ is)

def $(\Theta_n, S_n) = (\Theta\Theta_{n-1}, (S_{n-1})_{\Theta})$ // start with $*$, localize nothing



Eckman-Hilton argument: in strict cat, composition same

In weak cat, not same, but homotopic: gives you a Sphere's (S^n)-worth of morphisms.

$\boxed{(\infty,n)\text{-cats are the } sPSh(\Theta_n)_{\text{localised}}}$

4/4

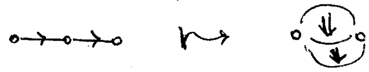
Lecture 7 (∞,n)-cat to be a fibrant object in $sPSh(\Theta_n)^{inj}_{S_n}$ 02/09/12

"Θ_n-spaces" cartesian presentation

Regard Θ_n as a strict n-cat, & via Yoneda, as $\Theta_n \subseteq sPSh(\Theta_n)$ ~~via the~~

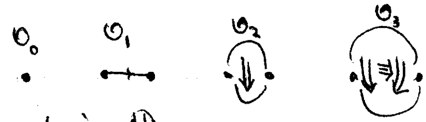
inclusion functor $\tau : \Theta_n \hookrightarrow \Theta_{n+1}$ ("no non-identity $n{+}1$-morphisms")

suspension functor $\sigma : \Theta_n \to \Theta_{n+1}$ $\sigma(\theta) = [1](\theta)$



$\sigma^k, \tau^k : \Theta_{n-k} \to \Theta_n$.

put $\mathcal{O}_k = \sigma^k[0]$ "free k-morphism"



etc...

& using τ^i can consider this object in Θ_n, $n \geq k$.

We have $\delta^0, \delta^1 : [0] \to [1]$ in Δ

iterating $s_k := \sigma^{k-1}\delta_0$, $t_k = \sigma^{k-1}\delta_1 : \mathcal{O}_{k-1} \to \mathcal{O}_k$

"source & target of $k-1$ cells"

1/5

put $\partial\Theta_k$ = subcat of Θ_k not containing k-morphism $\sigma^{k-1}(*)$

$\partial\Theta_k$: $\emptyset$ 

$e_k : \partial\Theta_k \longrightarrow \Theta_k$

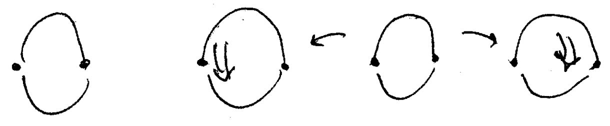
"pair of // k-1 morphisms"

exercise: (1) Show

$$\begin{tikzcd} \operatorname{colim}(\Theta_{k-1} & \partial\Theta_{k-1} \arrow[l] \arrow[r] & \Theta_{k-1}) \\ & \partial\Theta_k \arrow[from=1-1, "s_k"] \arrow[from=1-3, "t_k"'] & \end{tikzcd}$$

$\cong$

(ie. $S^k = D^k \amalg_{S^{k-1}} D^k$)



$X \in sPSh(\Theta_n)$ is Segal fibrant ↙ "space of k cells" in X

$\Gamma(X^{\Theta_k})$ "global sections"

write $\overline{X}(\Theta_k) = \mathrm{Maps}_{sPSh(\Theta_n)}(\Theta_k, X) \in Sp$

$\overline{X}(\partial\Theta_k) = \mathrm{Maps}_{sPSh(\Theta_n)}(\partial\Theta_k, X) \in Sp$ "space of pairs of // k-1 cells in X"

Segal condition $\Rightarrow$ $\overline{X}(\partial\Theta_k) \xrightarrow{\simeq} \overline{X}(\Theta_{k-1}) \times_{\overline{X}(\partial\Theta_{k-1})} \overline{X}(\Theta_{k-1})$

given $(f_0, f_1) \in \overline{X}(\Theta_{k-1}) \times \overline{X}(\Theta_{k-1})$

write $\underline{\mathrm{Map}}_X(f_0, f_1)$ for the $sPSh(\Theta_{n-k})$

"$\lim(\{(f_0,f_1)\} \longrightarrow \overline{X}(\partial\Theta_k))$"

$\left(\Theta \in \Theta_{n-k} \longmapsto \underleftarrow{\lim}(\overline{X}(V(1)^k F\Theta) \longrightarrow \overline{X}(V(1)^k\emptyset)) \right)$ (F is Yoneda embed.)

Θ corresponds to $k+1, \ldots, n$-morphisms $\longrightarrow$ // $\left[\underline{\mathrm{Map}}_X(f_0,f_1)(\Theta) = \mathrm{hofiber}_{(f_0,f_1)}\left(X(\sigma^k(\Theta)) \longrightarrow X(\Theta_{k-1})\right) \right]$

Immediate that (i) X Segal fibrant $\Rightarrow$ $\underline{\mathrm{Map}}_X(f_0,f_1)$ Segal fibrant in $sPSh(\Theta_{n-k})$

(ii) X Segal + complete fibrant $\Rightarrow$ ——— Segal + complete fibrant

requires small proof identifying what complete at level k, $\forall k$ is.

$\boxed{\text{ie., if } X \text{ is a } \Theta_n\text{-space, } \underline{\mathrm{Map}}_X(f_0,f_1) \text{ is } \Theta_{n-k} \text{ space.}}$

def An (∞,n)-cat $\overline{X}(\Theta_k)$ is contractible for $k<d$

(= fibrant $X \in sPSh(\Theta_n)^{inj}_{S_n}$) is called a "E_d-monoidal $(\infty, n-d)$-cat"

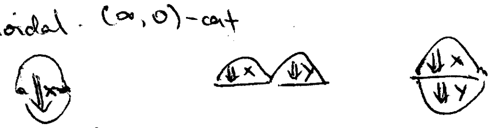
$d \le n$

Write $\overline{X}(\Theta_k) \sim *$

So take $* \in \overline{X}(\Theta_{d-1})$, $\underline{\mathrm{Map}}_X(*,*)$ is an $(\infty, n-d)$-cat.

But it still has d ~~extra~~ multiplication maps, which satisfy various relations we'll investigate.

e.g. $d=2=n$, E_2-monoidal $(\infty,0)$-cat



$x \in \overline{X}(\Theta_2)$

We'll describe E_d-operad $\Omega^d : E_d$-monoidal $(\infty,n-d)$-cat $\longrightarrow$ algebras for E_d operad in $(\infty,n-d)$-spaces $: B^d$

~~example~~ <u>defn</u> : a "monoidal (∞,n)-cat" is an E_1-monoidal $(\infty,n+1)$-cat

ie an $(\infty,n+1)$-cat X st. $\bar{X}(0_\bullet) \sim *$

<u>exercise</u> put $\mathcal{C} = \Omega X = \underline{\mathrm{Map}}_X(*,*)$

$$\Omega X \times \Omega X \longrightarrow \Omega X$$

the additional multiplication $(\Downarrow A) \times (\Downarrow B) \rightsquigarrow (\Downarrow C)$

can be interpreted as $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

& conversely, given such $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ satisfying some extra conditions we can recover $X = B\mathcal{C}$.

<u>exercise</u> Write out precisely what structure $\otimes$ has ("homotopy assoc") + def of B.

§ Fully dualizable objects, after Lurie

<u>def</u> $\mathcal{E}$ is a 2-cat, $X, Y \in \mathrm{ob}\,\mathcal{E}$, $f : X \to Y$, $g : Y \to X$

a 2-morphism $u : 1_X \to gf$ is "the unit of an adjunction"

if $\exists$ 2-morphism $v : fg \to 1_Y$ "counit of adjunction"

s.t. (i) $f = f \circ 1_X \xrightarrow[1 \times u]{} f \cdot (g \cdot f) = (f \cdot g) \cdot f \xrightarrow[v \times 1]{} 1_Y \cdot f = f$

is the identity

(ii) $g = 1_X \circ g \xrightarrow{u \times 1} (g \cdot f) \cdot g = g \cdot (f \cdot g) \xrightarrow{1 \times v} g \cdot 1_Y = g$

is the identity.

We say f is left adjoint to g.

<u>Lem</u> If $v : fg \to 1_Y$, $v' : fg \to 1_Y$

has v satisfies (i), v' satisfies (ii)

then $v = v'$. In particular, either of (i) or (ii) uniquely determines v.

<u>Pf</u> easy (see first lecture)

<u>eg.</u> (i) $\mathcal{E} = \mathrm{Cat}$, usual notion of adjoint functor

(ii) if $(\mathcal{C}, \otimes)$ a monoidal cat,

let $B\mathcal{C}$ 2-cat, with one object $*$, $\mathrm{Mor}(*,*) = \mathcal{C}$

composition of morphisms is $\otimes$.

then $X \in \mathcal{C}$, thought of as 1-morphism in $B\mathcal{C}$ has a right adjoint $Y \in \mathcal{C}$ $\Leftrightarrow$ Y right dual to X in $(\mathcal{C}, \otimes)$.

<u>eg.</u> $\mathcal{C} = (\mathrm{Vect}_k, \otimes)$ V has right adjoint $\Leftrightarrow$ V is f.d.

$\boxed{\text{having an adjoint is a generalization of f.d.}}$

(iii) If $f: X \to Y$ is invertible, with inverse $g: Y \to X$
$1_X \xrightarrow{\sim} gf$, $fg \xrightarrow{\sim} 1_Y$ so g adjoint to f.

Conversely, if u, v are isos, then f, g invertible.
So in particular, if every 2-morphism is invertible, then having an adjoint $\Leftrightarrow$ invertible. (*)

<u>def</u>: (i) a 2-Cat $\mathcal{E}$ has "adjoints for 1-morphisms" if every $f: X \to Y$ in $\mathcal{E}$ has both a left and right adjoint.
(ii) an (∞, n)-cat $\mathcal{E}$ has adjoints for 1-morphisms if its associated homotopy 2-cat $ho_2(\mathcal{E})$ does. // ho_2 takes Ho <u>Maps</u>(f_0, f_1)?

<u>eg.</u> If every 2-morphism in $ho_2(\mathcal{E})$ is invertible, then it admits adjoints for 1-morphisms $\Leftrightarrow$ every 1-morphism invertible $\Leftrightarrow$ $ho_2(\mathcal{E})$ is a groupoid.

<u>def</u> $\mathcal{C}$ is an (∞, n)-cat
(i) If $1 < k < n$, $\mathcal{C}$ admits "adjoints for k-morphisms" if $\forall X, Y \in ob\,\mathcal{C}$, the $(\infty, n-1)$-cat <u>Maps</u>(X, Y) admits adjoints for $k-1$ morphisms.
(ii) $\mathcal{C}$ "admits adjoints" if it admits adjoints for k-morphisms $\forall 0 < k < n$.

If every $k+1$-morph is invertible, adjoints for k-morphisms $\Leftrightarrow$ every k-morphism invertible also.
In particular, if $\mathcal{C}$ admits adjoints for n-morphisms also, then every k-morphism is invertible $\forall k \geq 0$.
i.e., $\mathcal{C}$ is an $(\infty, 0)$-cat, ie. a space (∞-groupoid)

<u>eg.</u> If $(\mathcal{C}, \otimes)$ is a monoidal (∞, n)-cat
say "$\mathcal{C}$ admits duals" if $B\mathcal{C}$ admits adjoints.
(ie: (i) $\mathcal{C}$ admits adjoints & (ii) In $(ho(\mathcal{C}), \otimes)$ every object has a dual.)

<u>prop:</u> $\mathcal{C}$ [sym] monoidal (∞, n)-cat (extra structure)
$\exists$ a symmetric monoidal (∞, n)-cat, $\mathcal{C}^{fd}$ which admits duals, & sym monoidal functor $\mathcal{C}^{fd} \to \mathcal{C}$ st.
any sym monoidal functor $\mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ admits duals, factors through $\mathcal{C}^{fd}$
(throw away ~~objects~~ k-morphisms w/o adjoints, starting at $k=1$)

<u>def</u> $X \in \mathcal{C}$ is "fully dualizable" if its in the essential image of $\mathcal{C}^{fd}$.

§ Lurie's Thm

M an m-manifold, $m \le n$. An "n-framing of M" is a trivialization $T_M \oplus \mathbb{R}^{n-m} \xrightarrow{\sim} \mathbb{R}^n$

Note $O_n(\mathbb{R})$ acts on $\mathbb{R}^n$, hence on n-framings.

<u>thm</u>/VERY rough form: $\exists$ an (∞,n)-cat Bord_n^{fr}

objects: framed 0 manifolds (ie. disjoint bunch of elements of $O_n(\mathbb{R})$)

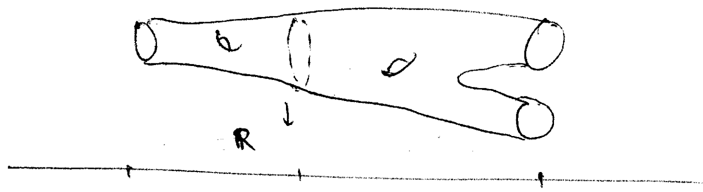
morphisms: framed bordisms between 0-manifolds

2- " ---------- framed bordisms between 0-manifolds

⋮

n-morphisms: framed n-manifolds (with lots of corners)

$\rightsquigarrow$ can make this a Segal cat, easily



Sym monoid functor from this cat to another completely determined by pt.

... really vague: circle action, Hodge theory, $\boxed{5/5}$

$\boxed{\text{Lecture 8}}$ 02/16/12

If $\mathcal{C}, \mathcal{D}$ are sym monoidal (∞,n)-cats, write

$\mathrm{Funct}^{\otimes}(\mathcal{C},\mathcal{D})$ for the (∞,n)-cat of sym monoidal functors $\mathcal{C} \to \mathcal{D}$

(If we ignore "symmetric", a monoidal (∞,n)-cat $\mathcal{C}$ is really a Θ_{n+1}-space $X = B\mathcal{C}$ with $X(\mathcal{O}_0) \sim *$ contractible

so if $\mathcal{D}$ is another such, $M = B\mathcal{D}^{B\mathcal{C}}$ is an $(\infty, n+1)$ cat with contractible $M(\mathcal{O}_0)$)

so we're OK here.

<u>thm</u> (Lurie) "Baez-Dolan cobord hypothesis"

let $* = (pt, \mathbb{R}^n)$ be the standard framed point

Then Bord_n^{fr} is the sym. monoidal (∞,n)-cat with duals freely generated by the object $*$.

ie., if $\mathcal{C}$ is an (∞,n) sym monoidal -cat. with duals, then

$Z \in \mathrm{Funct}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C})$ is an (∞,n). cat in which

all k-morphisms are invertible, for all k, ie. an $(\infty,0)$-cat, ie, a space.

map $Z \mapsto Z(*)$ gives an equivalence

$\mathrm{Funct}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \xrightarrow{\sim} \tilde{\mathcal{C}}$

where $\tilde{\mathcal{C}}$ is the $(\infty,0)$-cat obtained from $\mathcal{C}$ by discarding all non-invertible k-morphisms $\forall k$.

$\boxed{1/4}$

★ exercise: If $\mathcal{C}$ is a Θ_n-space, what is the def of $\tilde{\mathcal{C}}$? // just restrict sPSh to Θ_0 to get a space = sSet = $(\infty,0)$-cat

"extended d-TFTs are determined by their value on a pt"

(e.g. $d=2$, $\mathcal{C}=\mathrm{Vect}_k$ 2-TFT: Frob alg
$\mathcal{C}=\mathrm{Ch}(k)$ Kontsevich, Costello)

Generalization, after Baez-Dolen, "Tangle hypothesis"

Lurie: $0 \le k \le n$, $m \le n$, V an m-framed n-manifold

$$\varphi: T_V \times \mathbb{R}^{n-m} \xrightarrow{\sim} \mathbb{R}^n$$ // of tangent bundles

def: a "k-framed submanifold of V" is a pair (M,g)

where (i) M is a submanifold of V, codim $M = n-k$ ($\Rightarrow \dim M = m-n+k$)

so $T_M \times \mathbb{R}^{n-m} \subseteq T_V|_M \times \mathbb{R}^{n-m} \xrightarrow[\varphi]{\sim} \mathbb{R}^n$ subbundle

gives a section of the trivial $\mathrm{Grass}_{n-k}(\mathbb{R}^n)$-bundle on M, ie a map $\sigma: M \longrightarrow \mathrm{Gr}_{n-k}(\mathbb{R}^n) = O_n\mathbb{R}/O_k\mathbb{R} \times O_{n-k}\mathbb{R}$.

(ii) g is a homotopy from σ to a constant map $M \longrightarrow * \in \mathrm{Grass}_{n-k}(\mathbb{R}^n)$

(if V has boundary/corners, requires that ∂V/corners intersect M transversely.)

Lurie, thm: $\exists$ an (∞,k)-cat, $\mathrm{Tang}^V_{(k,n)}$ with objects k-framed compact submanifolds of V

morphisms $M_0 \longrightarrow M_1$ are k framed submanifolds $\tilde{M}$ of $V \times [0,1]$

st. $\tilde{M} \cap V \times \{i\} = M_i$, $i=0,1$ "and so on..."

let $D_r = \{x \in \mathbb{R}^r \mid |x| < 1\}$ open disc in $\mathbb{R}^r$ centered at 0.

define $\mathrm{Tang}_{k,n} = \mathrm{Tang}^{D_{n-k}}_{k,n}$

embeddings $\mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^{n-k+1} \hookrightarrow \cdots$

induce $\mathrm{Tang}_{k,n} \hookrightarrow \mathrm{Tang}_{k,n+1} \hookrightarrow \cdots$ of (∞,k)-cats,

& $\varinjlim_n \mathrm{Tang}_{k,n} = \mathrm{Bord}^{fr}_k$, as data of ($k$-framed submanifold of $\mathbb{R}^\infty$) $\downarrow$ k-framed manifolds

has contractible fiber

$\mathrm{Tang}_{k,n}$ ~~not~~ symmetric monoidal, but does naturally carry an action of E_{n-k}-operad ("operad of little discs") & so is naturally an E_{n-k}-monoidal (∞,k)-cat

... given an embedding of α-disjoint discs, $\alpha \in \mathbb{N}$

$$D \sqcup D \sqcup \cdots \sqcup D \hookrightarrow D$$

get an (∞,k)-functor $\mathrm{Tang}_{n,k} \times \cdots \times \mathrm{Tang}_{n,k} \longrightarrow \mathrm{Tang}_{n,k}$

Lurie's thm "$\mathrm{Tang}_{k,n}$ is the E_{n-k}-monoidal (∞,k)-cat with duals freely generated on one object" (warning: if $k=n$, slightly finicky bits in the def. S^0 is not connected)

ie, if $\mathcal{C}$ is an E_{n-k}-monoidal (∞,k)-cat w/ duals,

let $* \in \mathrm{ob}\,\mathrm{Tang}_{k,n}$ be $\{0\} \subseteq \mathbb{R}^{n-k}$ as a std framed manifold,

then evaluation at $(*)$ gives an equiv of $(\infty,0)$-spaces $\mathrm{Funct}^{\otimes}(\mathrm{Tang}_{k,n}, \mathcal{C}) \to \tilde{\mathcal{C}}$

Cor: as $\mathrm{Bord}_n^{fr} = \varinjlim_d \mathrm{Tang}_{k,k+d}$

put $E_\infty = \varinjlim_d E_d$,

& "sym. monoidal" now means "E_∞-monoidal"

& this thm implies previous cobord hyp.

This is all so we can avoid defining symmetric...

thm (i) Let X be an E_d-monoidal (∞,k)-cat with duals, freely generated by an object $*$. Then X admits an $O_d\mathbb{R} \times O_k\mathbb{R}$-action.

(ii) [Lurie] $X \longrightarrow \mathrm{Tang}_{k,d+k}$ equiv as E_d-monoidal (∞,k)-cats

this is all vague, details in next lecture

Recall: $X \in \Theta_{k+d}$-spaces st. $\bar{X}(O_r) \sim *$ if $r<d$ is an "E_d-monoidal (∞,k)-cat"

two special cases: (i) $d=0$, We have E_0-monoidal (∞,k)-cat, ie. Θ_k-space generated by one obj, $*$, with adjoints for t-morphisms for all $t<k$.

(ii) $k=0$, E_d-monoidal $(\infty,0)$-cat, & no condition on duals at all.

classically, such a thing is exactly $\Omega^d Y$, $Y \in Sp$, $\Omega^d = \mathrm{Maps}(S^d, \cdot)$

Thm of May [Segal, Thomason].

$$A \text{ a dga}, \quad Z(S^1) = A \overset{L}{\underset{A\otimes A^{op}}{\otimes}} A \underset{HKR}{=} \Lambda^{\cdot}\Omega^1_A \qquad A = k[X],\ X \text{ affine, smooth}$$

0-differential

Hochschild, ..., Rosenberg

// Lurie's thm ⇝ circle action ("$\cong O_d$") gives de Rham differential --- conceptual reason

---

$\varphi: X \longrightarrow Y$ $\quad$ G acts on X, Y, φ is a G-map

(i) declare φ a weak equiv if $\varphi: X \to Y$ is a weak homotopy equiv forgetting G-action

(ii) declare φ a ______ if $\varphi: X^H \longrightarrow Y^H$ is ______ for all $H \subseteq G$

"strong G-equiv"

(iii) $\mathcal{F} \subseteq \{\text{subgps of } G\}$ be closed under conjugation & inclusion.

$\mathcal{F}$-equiv if $\varphi: X^H \longrightarrow Y^H$ w.e. $\forall H \in \mathcal{F}$

Slogan: "equivariant homotopy type is a presheaf on discrete cat of subgps of G"

We can consider restriction to subcat --→ how small can we get?

Hochschild ~~coh~~ homology ⇝ $k[x]$ gives $k[x, dx]$ de Rham coh over $\mathbb{Q}$

(tangent sp to K-theory) over $\mathbb{F}_p$ --- $\mathbb{F}_p[x, dx]$ doesn't work so well (need p-adic Hodge theory)

When not over $\mathbb{Q}$:

Topological Hochschild / cyclic homology ⇝ don't use all of circle action

↳ finite subgps

- J.F. Adams, Stable Homotopy Theory — Ref for Spectra

  Cdont read defn of products on homotopy cat; 20 yrs ago, Hopkins discovered better way to define "upstairs" — S-modules.

Spectra = "spaces w/ suspension inverted."

02/21/12

Lecture 9

define, for $m \geq 0$, sym. monoidal $(\infty,1)$-cat $m\text{-Alg}$

———— $(\infty, m+1)$-cat $m\text{-Alg}^{\text{Morita}}$

———— (∞, m)-cat $m\text{-Alg}_0^{\text{Mor}}$

this last will be: take $m\text{-Alg}^{\text{Mor}}$, & throw away non-invertible $m{+}1$-morphisms.

<u>$m=0$</u>: $0\text{-Alg}^{\text{Mor}} := (Ch(k), \overset{L}{\otimes})$ sym. monoidal

regard this as a CSS (= $(\infty,1)$-cat) via $(Ch(k), qi) \rightsquigarrow \mathcal{N}_{qi}\, Ch(k)$

Its very computable, as lots of Quillen equiv model cat structures on it & you can compute this CSS easily.

If $X \in Ch(k)$, set $X^* = Hom_{Ch(k)}(X,k)$

X dualizable if $\exists$ morphism $\underset{k}{\overset{\shortparallel}{1}} \longrightarrow X^* \otimes X$ (as always have $X \otimes X^* \to 1$) $\Leftrightarrow \sum \dim H^i(X) < \infty$

Set $0\text{-Alg} = \{(X,x) \mid X \in Ch(k),\ x: 1 \to X$ a chain map $\}$
$x \in Z^0(X)$

<u>$m=1$</u>: 1-Alg (= dgCat)

objects of $1\text{-Alg} = ob(1\text{-Alg}^{\text{Mor}}) =$ dga's A/k

$1\text{-Alg}(A,B) = \{F: A \to B \mid F$ homo of dga's$\}$

compose by $\otimes$ $\rightsquigarrow$ $1\text{-Alg}^{\text{Mor}}(A,B) = \{M \in {}_A\text{Mod}_B \mid$ this is a dgCat, & so an $(\infty,1)$-cat$\}$

chain complexes of bimodules

${}_A\text{Mod}_B(M,N) = \{\varphi: M \to N$ chain complex morphism of bimodules$\}$

"intertwines"

$1\text{-Alg}_0^{\text{Mor}}$: $(\infty,1)$-cat where throw away non-invertible intertwiners

Every $A \in 1\text{-Alg}_0^{\text{Mor}}$ is dualisable, with dual A^{op}

$ev: A \otimes A^{op} \xrightarrow[A]{} 1$, $coev: 1 \xrightarrow[A]{} A^{op} \otimes A$

compose: $A \xrightarrow[A]{} A \otimes 1 \xrightarrow[A \otimes A]{} A \otimes A^{op} \otimes 1 \xrightarrow[A \otimes A]{} 1 \otimes A \xrightarrow[A]{} A$

check that $A \underset{A}{\otimes} (A \otimes A) \underset{A \otimes A^{op} \otimes A}{\otimes} (A \otimes A) \underset{A}{\otimes} A \xrightarrow{\sim} A$

& similarly for other direction: $A^{op} \xrightarrow[A]{} A^{op} \otimes A \otimes A^{op} \xrightarrow[A]{} A$

$1\text{-Alg}^{\text{Mor}}$ is an $(\infty,2)$-cat, so as well as above, we now impose the condition there are adjoints for 1-morphisms

For example, $ev: A\otimes A^{op} \xrightarrow[A]{} \mathbb{1}$ must admit left & right adjoints.

prop: A is fully dualizable $\Leftrightarrow$ this morphism $A \in {}_{A\otimes A^{op}}\mathrm{Bimod}_{\mathbb{1}}$ has both left & right adjoints

$\Leftrightarrow$ (i) $\sum \dim H^i(A) < \infty$ & (ii) $A \in \mathrm{Perf}(A\otimes A^{op})$ ← finite resolutions by projections

A is "proper" A is "smooth" ---- " Jacobian crit for smoothness via cotangent bundle being v.b. (diagonal) "

$\Leftrightarrow$ (i) A is dualizable in Ch(k) (ii) A is dualizable in $A\otimes A^{op}$-mod.

examples: $A \in \mathrm{Vect}_k$ (i.e. $H^i(A)=0$, $i\neq 0$)

A fully dualizable $\Leftrightarrow$ A is f.d. & semisimple "separable"

Let $\mathcal{S}$ be a sym. monoidal $(\infty,1)$-cat

& for $m\geq 1$ define $m\text{-}\mathrm{Alg}(\mathcal{S}) = \mathrm{Alg}((m-1)\text{-}\mathrm{Alg}(\mathcal{S}))$

the $(\infty,1)$-cat of Alg. objects on the sym monoidal $(\infty,1)$-cat $(m-1)\text{-}\mathrm{Alg}(\mathcal{S})$

$m\text{-}\mathrm{Alg}^{\mathrm{Mor}}(\mathcal{S})$: objects = ob of $m\text{-}\mathrm{Alg}(\mathcal{S})$

$\mathrm{Maps}_{m\text{-}\mathrm{Alg}^{\mathrm{Mor}}}(A,B) = (m-1)\text{-}\mathrm{Alg}^{\mathrm{Mor}}({}_A\mathrm{Mod}_B)$

where ${}_A\mathrm{Mod}_B$ is ~~actually~~ naturally in $(m-1)\text{-}\mathrm{Alg}(\mathcal{S})$

So $m\text{-}\mathrm{Alg}^{\mathrm{Mor}}(\mathrm{Ch}(k))$ is a "higher version" of dgCat.

$(\mathcal{C},\otimes)$ sym monoidal $(\infty,1)$ cat, $X\in\mathcal{C}$ dualizable, $\mathbb{1}$

$\mathbb{1} \xrightarrow{coev} X\otimes X^* \xrightarrow{ev} \mathbb{1}$ call composite "$\dim X$" $\in \mathrm{Map}_{\mathcal{C}}(\mathbb{1},\mathbb{1})$

If $\mathcal{C} = 1\text{-}\mathrm{Alg}_0^{\mathrm{Mor}}$, $A\in\mathcal{C}$ is a dga

$\mathbb{1} \xrightarrow[A]{} A\otimes A^{op} \xrightarrow[A]{} \mathbb{1}$ "$\dim A$" $= A \overset{L}{\underset{A\otimes A^{op}}{\otimes}} A$ Hochschild chains.

Cobord hyp gives $Z: \mathrm{Bord}_1^{Fr} \to \mathcal{C}$ $\phi \; C^{X}_{X^\vee} \supset \phi$

so "$\dim X$" $= Z(S^1)$

But $\mathrm{Maps}_{\mathrm{Bord}}(\phi,\phi) \in (\infty,0)\text{-}\mathrm{Cat} = \mathrm{Sp}$ "is the classifying space for 1-dim oriented closed manifolds $\coprod_{k\geq 0} (S^1)^{\amalg k}$"

Point is that "$\dim X$" has an S^1-action.

In our example, that $A \overset{L}{\underset{A\otimes A^{op}}{\otimes}} A$ is a module for $C_*(S^1)$

= Kahler differentials

§ Geometric realization

Classically, have $|\cdot| : \Delta^{op}Set \rightleftarrows Top : Sing$

comes from a cosimplicial space $\mathbb{\Delta} \in Funct(\Delta, Top)$

$$n \mapsto \{(x_i) \in \mathbb{R}^{n+1} \mid \sum_0^n x_i = 1\} = \mathbb{\Delta}_n$$

Idea is: every $X \in \Delta^{op}Set$ is built out of Δ_n's by glueing, i.e. as a colim

& $|\cdot|$ left adjoint, so idea is set $|\Delta_n| = \mathbb{\Delta}_n$

& build $|X|$ out of $\mathbb{\Delta}_n$'s the way you build X out of Δ_n.

explicitly, formally

$$\mathrm{coeq}\Big(\coprod_{\alpha:[n]\to[m]} X_m \times \mathbb{\Delta}_n \underset{\mathbb{\Delta}(\alpha)}{\overset{X(\alpha)}{\rightrightarrows}} \coprod_n X_n \times \mathbb{\Delta}_n \Big) \longrightarrow |X|$$

note $|X \times Y| \longrightarrow |X| \times |Y|$ homotopy equiv

this extends to $|\,| : \Delta^{op} Top \rightleftarrows Top$ by exactly the same formula.

now $X_n \times \mathbb{\Delta}_m$ is a product of Top. spaces,

before X_n was a discrete top space.

Let's replace Top by Sp.

the classical context for this "Reedy Cat"

thm: Let $\mathcal{C}$ be a simplicial model category, for example Sp.

Then there exists a model cat str on $\Delta^{op}\mathcal{C}$ "Reedy str"

st. $|\cdot| : \Delta^{op}\mathcal{C} \rightleftarrows \mathcal{C} : (\ )^{\Delta}$ $\qquad X \in \mathcal{C}, \ X^{\Delta} \in \Delta^{op}\mathcal{C}$, $n \mapsto X^{\Delta^n}$

are Quillen adjoint functors

$|\cdot|$ is given by exactly same formula (replace $\times$ by $\otimes_{\mathcal{C}}$).

- weak equivs are level wise, $f : X \to Y$
- cofib if $L_n^X Y = X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$ cofib in $\mathcal{C}$, $n \geq 0$
- fib if $X_n \longrightarrow Y_n \cup_{M_n Y} M_n X$ fib in $\mathcal{C}$, $n \geq 0$.

$$L_n X = \operatorname*{colim}_{\substack{\phi:[n]\to[k],\, surj \\ \phi \text{ not identity}}} X_k \qquad\qquad M_n X = \lim_{\substack{\phi:[k]\to[n] \\ \phi \text{ inj, not id}}} X_k$$

essential ingredient is: Δ is a "Reedy Cat"

def: a Reedy cat $\mathcal{R}$ is a small cat with two wide subcats $\mathcal{R}^+, \mathcal{R}^-$ "direct" "inverse"

contains all objects

a fn $\deg : ob\mathcal{R} \to \mathbb{N}$ st.

(i) every morphism in $\mathcal{R}$ factors uniquely, $\alpha = \alpha^+ \alpha^-$ with $\alpha^+ \in \mathcal{R}^+$, $\alpha^- \in \mathcal{R}^-$

(ii) if $\alpha : c \to d$ is in $\mathcal{R}^+$, $\deg(c) \leq \deg(d)$, equality $\Leftrightarrow$ α is identity map

$\alpha : c \to d$ in $\mathcal{R}^-$, then $\deg(c) \geq \deg(d)$, equality $\Leftrightarrow$ α identity

so $\mathcal{R}^+ \cap \mathcal{R}^- = \{\text{identity maps}\}$, & these are only iso's in $\mathcal{C}$.

example: Δ; $\deg [n] = n$, $\Delta^- =$ surjective maps
$\Delta^+ =$ inj maps.

Now generalize to $sPSh(\Theta_n)$.

Want to define $|\cdot| : \Theta_n \longrightarrow Sp$

s.t. $O_n \longmapsto B_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid \sum t_i^2 \leq 1\}$

$\partial O_n \longmapsto \partial B_n$

& get everything else by gluing, generalizing the cosimplicial simplex $\Delta \in \Delta(Top)$

then define $|\cdot|_{\Theta_n} : PSh(\Theta_n) \longrightarrow Sp$

by sending • Yoneda $F\theta \longmapsto |\theta|$ [need functor on Θ_n not just O_n]

• everything else by glueing (colimits)

Then extend to $sPSh(\Theta_n) \longrightarrow Sp$ by glueing.

& want this is sensible, ~~eg.~~ $|\cdot|_{\Theta_n}$ preserves products. 4/4

Lecture 10 Everything today in two papers of Clemens Berger, (2002, 2006) 02/23/12

thm: (1) There exists a co-Θ_n-space, ie. a functor $Disk : \Theta_n \longrightarrow Sp$ and ~~hence~~ adjoint functors

$$|\ |_{\Theta_n} : sPSh(\Theta_n) \rightleftarrows Sp : (\ )^{Disk}$$

where

$$|X|_{\Theta_n} = X \underset{\Theta_n}{\otimes} Disk$$

$$= coeq\left(\coprod_{\substack{\alpha : \theta \to \theta' \\ \text{in } \Theta_n}} X(\theta') \times Disk(\theta) \underset{Disk(\alpha)}{\overset{X(\alpha)}{\rightrightarrows}} \coprod_{\theta \in ob\,\Theta_n} X(\theta) \times Disk(\theta) \right)$$

If $Y \in Sp$, $Y^{Disk} : \theta \longmapsto Y^{Disk(\theta)} \in Sp$

so $Y^{Disk} \in sPSh(\Theta_n)$

st. (i) $|O_k|_{\Theta_n}$ "is" a k-Disk D_k

$|\Delta_m|_{\Theta_n} = \Delta_m$ $\quad [m]$: $\bullet\!\to\!\bullet\!\to\!\bullet\!\to\!\bullet \cdots \bullet\!\to\!\bullet$ $\quad \Delta_m = F_{[m]} = \Delta(\cdot, [m])$

(ii) $|\ |_{\Theta_n}$ preserves finite limits, in particular, the natural map $|X \times Y| \longrightarrow |X| \times |Y|$ is a weak equiv

(iii) Θ_n is a Reedy cat.

Variant: $Disk^{top} : \Theta_n \longrightarrow Top$ sends $|O_n| = D_n$

& $|X \times Y|^{top} \longrightarrow |X|^{top} \times |Y|^{top}$ is a homeo

& there is a non-degen cell of dim $\deg \theta$ for every non-degen cell in X of type θ.

(Reedy str)

pf sketch #1: Define Disk(Θ) explicitly

- Show $|\cdot|_{\Theta_n}$ preserves finite limits

can do this explicitly, by writing $F_\Theta \times F_{\Theta'}$ explicitly as a colimit of cells $F_{\Theta''}$ & check by hand. This explicit def also shows what non-degen cells look like.

[Joyal, Berger]

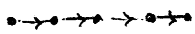
explicit combinatorics:

$\Theta_n \longleftrightarrow$ "planar level trees" of ht $\leq n$

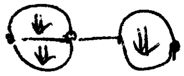
$\theta \longleftrightarrow T_\theta$

Θ_1 $\quad [0] \longmapsto \bullet$

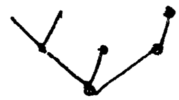
$[n] \longmapsto T_{[n]}$



Θ_2

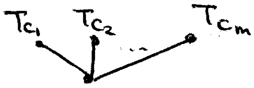


$\rightsquigarrow$



$[3]([2],[0],[1])$

ie inductively, $[m](c_1,\dots,c_m) \longmapsto$

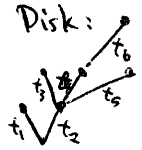


So $\Theta_n \longmapsto$

Put $\deg[m](c_1,\dots,c_m) = m + \sum \deg(c_i)$ = # edges in the tree.

Explicit Disk:

(variant) in Top

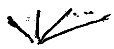


$\rightsquigarrow$ $\left\{ (t_1,\dots,t_6) \in [-1,1]^6 \;\middle|\; \begin{array}{l} t_1 \leq t_2, \\ t_3 \leq t_4 \leq t_5, \\ t_2^2+t_3^2 \leq 1, \quad t_2^2+t_4^2+t_6^2 \leq 1 \\ t_2^2+t_5^2 \leq 1 \end{array} \right\}$

linear order at each level crucial

paths from root

So $[m]$



$\rightsquigarrow -1 \leq t_1 \leq t_2 \leq \dots \leq t_m \leq 1 = \Delta_m$

Θ_n

$\rightsquigarrow \{t \in \mathbb{R}^n \mid \sum t_i^2 \leq 1\} = D_n$.

$F\Theta \times F\Theta'$ is a (colimit) union of cells F_ψ, $\bigcup_\psi F_\psi$ ← Yoneda

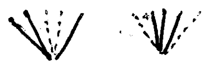
union over all ψ st. $T_\psi \in Sh(T_\theta, T_{\theta'})$

where $U \in Sh(T,T')$ if $T \cap T' = \{\text{root}\}$, $T \cup T' = U$
$(T, T' \subseteq U)$

"all ways to put trees together, keeping internal order, but allow permute between"

(so $\Delta_n \times \Delta_m$ is union of $\binom{m+n}{m}$ copies of Δ_{m+n})

e.g. for trees:



(dual picture: lattice paths)

critique: (1) to actually prove this works, must do some combinatorics in Θ_n.

What is its invariant meaning?

(2) Where does Disk(θ) come from, why is Disk(Θ_n) a ball ???

pf sketch #2: [Berger, 2006]

use the suspension map $\Theta_n \longrightarrow \Theta_{n+1}$ to construct $|\cdot|_{\Theta_n}$

$$\theta \longmapsto [1](\theta)$$

define $\delta_{\mathcal{C}} : \Delta \times \mathcal{C} \longrightarrow \Delta \wr \mathcal{C} = \Theta\mathcal{C}$

$$([n], A) \longmapsto [n](A, A, \dots, A)$$

$$(\alpha : [m] \to [n],\ f : A \to B) \longmapsto (\alpha, \text{"f on each factor"})$$

Suppose $|\cdot|_{\mathcal{C}} : sPSh(\mathcal{C}) \longrightarrow \begin{cases} Top \\ Sp \end{cases}$

$\left\{ \begin{array}{l} \text{s.t. it is} \quad \bullet \text{ colimit preserving} \quad [\Leftrightarrow \text{is a left adjoint}] \\ \qquad\qquad\quad \bullet \text{ finite limit preserving} \end{array} \right.$

call such a thing a "realization functor"

example $|\cdot| : PSh(\Delta) \longrightarrow Sp$ identity functor.

Then $|\cdot|_{\Theta\mathcal{C}} =: |\ |_{\Delta\times\mathcal{C}} \circ \delta_{\mathcal{C}}^* : sPSh(\Theta\mathcal{C}) \to Sp$

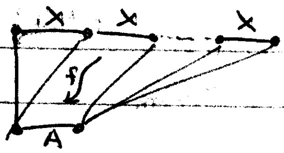
is also a realization functor.

pf: $|X \times Y|_{\Delta\times\mathcal{C}} =: |X|_\Delta \times |Y|_{\mathcal{C}}$ is obviously a realization functor &

its limit & colimit preserving (has left adjoint, denoted $(\delta_{\mathcal{C}})_!$ — left Kan extension

$\delta_{\mathcal{C}}^* : sPSh(\Theta\mathcal{C}) \longrightarrow sPSh(\Delta\times\mathcal{C})$ right adj ——— $(\delta_{\mathcal{C}})_*$ — right Kan extension

$$(\delta_{\mathcal{C}}^* F_{[1](A)})([n], X) = \Theta\mathcal{C}([n](X,\dots,X), [1](A))$$



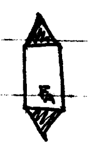
$$= \coprod_{\substack{\alpha : [n] \to [1] \\ \alpha \text{ surj}}} \mathcal{C}(X, A) \ \coprod_{\substack{\alpha : [n] \to [1] \\ \text{not surjective}}} *_\alpha$$

$\delta^{00\cdots 0}$

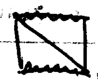
$\delta^{11\cdots 1}$

$\delta_{\mathcal{C}}^* F_{[1](A)} \in PSh(\Delta\times\mathcal{C})$ is $(\Delta^1 \times F_A = F_{[1]\times A})/\sim$

where $\delta^0(*) \times F_A \sim *_0$, $\delta^1(*) \times F_A \sim *_1$

Δ'  so, $|\delta_{\mathcal{C}}^* F_{[1](A)}| = \Delta' \times |F_A|/\sim$ as shown.

e.g. $\mathcal{C} = \Delta$, $A = [1]$, so $F_{[1][1]} = \Theta_2$

$\Delta' \times \Delta'$  so $|\Theta_2| =$ 

put $\delta_n : \Delta^n \xrightarrow[\delta_{\Delta^{n-1}}]{} \Delta \wr \Delta^{n-1} = \Theta \Delta^{n-1}$

$\downarrow \Theta \delta_{n-1}$

composite, $n \geq 2$ $\qquad \Theta \Theta_{n-1} = \Theta_n$

put $\delta_1 = \mathrm{Id} : \Delta \longrightarrow \Delta = \Theta_1$

defines $|\cdot|_{\Theta_n}$. it is colimit & finite limit preserving, for free

Define $\mathrm{Disk}(\Theta) = |F\Theta|_{\Theta_n}$ & adjointness clear

to identify non-degen cells in $|X|_{\Theta_n}$, must still show Reedy

<u>critique</u> still not clear why this works.

- what disks are?
- suspension def. is not intrinsic, it's extra structure

$\mathrm{Im}\, \delta_n$ = trees in which all nodes at ht k have fixed valence a_k

$\delta_n([a_0], [a_1], \ldots, [a_k])$ 

morally, this comes from n-cat = iterated multisimplicial set

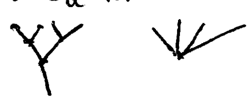
they suffice as test objects (right $\perp$ to them in Θ_n is the terminal objs)

<u>exercises</u> (i) compute $|\Theta_n|_{\Theta_n}$ & show it is disk + hemisphere decomp

(ii) compute $|F_{[n]}(A_1, \ldots, A_n)|$ in terms of $|FA_1|, \ldots, |FA_n|$

(iii) hence show this is homeo to previous sketch #1 geom realiz.

$(\Delta^1)^m \leftrightarrow \Delta^m$ Is it enough to build things in Θ_n from Θ_a for $a \leq n$?

<u>NO</u>. Eilenberg-Zilber / tree shuffling 

height not increasing by shuffling.

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02/28/12

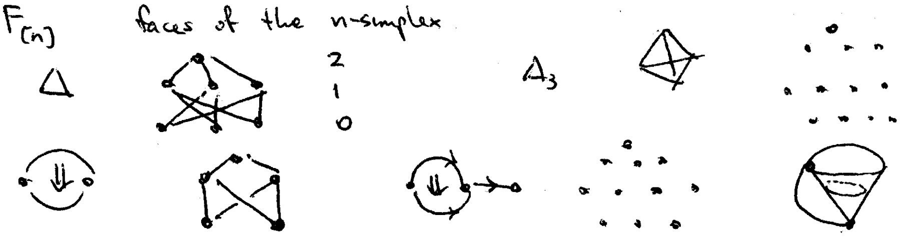
<u>Lecture 11</u>

pf sketch #3: If $\Theta \in \Theta_n$, look at the poset P_Θ of <u>non-degen</u> subobjects of Yoneda $F\Theta \in PSh(\Theta_n)$

example $\Delta_n = F_{[n]}$ faces of the n-simplex

Δ_2 $\qquad$ 2 1 0 $\qquad$ Δ_3

Θ_2



In each of these cases, this is always the face complex of a CW complex D

Moreover, D is the cone over a CW complex S & S is homeo to a sphere of dim deg Θ.

This is true in general! // we don't know what non-deg means yet // or how subobj of Θ relate to $F\Theta$

1/3

hence, as $N(P_\Theta)$ is the barycentric subdivision of D, hence homeo to D,

$\Theta \longmapsto N(P_\Theta)$ makes a good disk functor

homeo (but not equal) to the disk functors before

critique: <u>none</u>, ie once we show Θ_n is a Reedy cat + extra properties

(so, eg. can compute subobjects of $F\Theta$ in terms of Θ_n)

we have a purely internal def of $|\cdot|_{\Theta_n}$

Θ_n is a <u>good</u> Reedy cat

let $\mathcal{C}$ be a cat, $c; d_1, \dots, d_n \in \mathrm{ob}\,\mathcal{C}$

Write $\mathcal{C}(c; d_1, \dots, d_n) = \mathcal{C}(c, d_1) \times \dots \times \mathcal{C}(c, d_n)$ $n \geq 1$

$\mathcal{C}(c;) = *$ one point set if $n = 0$

this forms a "symmetric comulti-cat" $\mathcal{C}(*)$

"single input, multiple outputs"

- convenient notation for morphisms in $\Theta\mathcal{C} = \Delta \wr \mathcal{C}$

$\Theta\mathcal{C}([m](c_1, \dots, c_m), [n](d_1, \dots, d_n)) = \{ (\alpha : [m] \to [n], f_i) \}$ in Δ, $i = 1, \dots, m$

where $f_i = (f_{ij}) \in \mathcal{C}(c_i; d_{\alpha(i-1)+1}, \dots, d_{\alpha(i)})$

def: [Rezk-Bergner, after Berger] a comulti Reedy cat, is a cat $\mathcal{C}$, wide subcat $\mathcal{C}^- \subseteq \mathcal{C}$ "degeneracies", $\mathcal{C}^+(*) \subseteq \mathcal{C}(*)$ faces

$\deg : \mathrm{ob}\,\mathcal{C} \to \mathbb{N}$ st.

(i) every multi-morph $\alpha = (\alpha_s : c \to d_s)_{s=1, \dots, m}$

factors uniquely $\alpha = \alpha^+ \alpha^-$ with $\alpha^- : c \to x$ in $\mathcal{C}^-$

$\alpha^+ : x \to d_1, \dots, d_m$ in $\mathcal{C}^+(*)$

(ii) for every $\alpha : c \to d_1, \dots, d_n$ in $\mathcal{C}^+(*)$, $\deg(c) \leq \sum \deg(d_i)$,

moreover if $\alpha : c \to d$ is in $\mathcal{C}^+ := \mathcal{C}^+(*) \cap \mathcal{C}$, $\deg(c) = \deg(d)$

$\Leftrightarrow$ α is an identity map.

(iii) if $\alpha : c \to d$ is in $\mathcal{C}^-$, $\deg(c) \geq \deg(d)$

equality $\Leftrightarrow$ α is an identity map

Example: Δ, put $\deg[m] = m$, $\Delta^- = \{ \alpha : [n] \to [m] \text{ surj maps} \}$

$\Delta^+(*) = \{ \alpha_i : [m] \to [n_1], \dots, [n_m]$ st. $[m] \to [n_1] \times \dots \times [n_m]$ is injective,

$r \mapsto (\alpha_1(r), \dots, \alpha_n(r))$ $\}$

$\Leftrightarrow$ $\forall \beta, \beta' : [k] \to [m]$, if $\alpha_i \beta = \alpha_i \beta'$ $\forall i$, then $\beta = \beta'$

observe: $\Delta^+([m]; [n_1], \dots, [n_m])$ index the non-degen simplices in $\Delta^{n_1} \times \dots \times \Delta^{n_m}$

- Note $\mathcal{C}^+ := \mathcal{C}(*) \cap \mathcal{C}^+(*)$, then $\mathcal{C}, \mathcal{C}^+, \mathcal{C}^-, \deg$ is a usual Reedy cat.

lemma: $\mathcal{C}$ a comulti-Reedy cat $\Rightarrow \Theta\mathcal{C}$ is, where

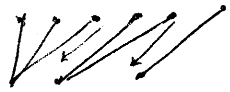
(i) $(\Theta\mathcal{C})^- = \{(\alpha, f) : [m](c_1, \dots, c_m) \longrightarrow [n](d_1, \dots, d_n)\}$

st. $\alpha \in \Delta^-([m],[n])$ is surjective

and if $\alpha(i-1) < \alpha(i)$, then $f_i : c_i \longrightarrow d_{\alpha(i)}$ is in $\mathcal{C}^-$

(ii) $(\Theta\mathcal{C})^+(*) \ni \quad f = (f_1, \dots, f_N)$

$$f_s = (\alpha_s, (f_{si})) : [m](c_1, \dots, c_m) \longrightarrow [n_s](d_{s1}, \dots, d_{s,n_s})$$



st. (a) multimorph $\alpha_s : [m] \longrightarrow [n_1], \dots, [n_N]$ is in $\Delta^+(*)$

(b) for each i, multimorph $f_{sij} : c_i \longrightarrow d_j \Big|_{\substack{s=1,\dots,N \\ j=\alpha_s(i-1)+1,\dots,\alpha_s(i)}}$ is in $\mathcal{C}^+(*)$

(iii) $\deg [m](c_1, \dots, c_m) = m + \sum \deg c_i$.

pf: Straightforward. Must check:

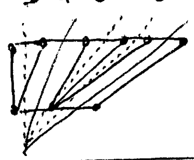
- all identity morphisms in $\Theta\mathcal{C}^-$, $\Theta\mathcal{C}^-$ closed under comp
- all identity morphs in $\Theta\mathcal{C}^+(*)$, & $\Theta\mathcal{C}^+(*)$ closed under multi-comp.
- every $f \in \Theta\mathcal{C}(*)$ factors uniquely
- deg condition on multi-morphs

$\square$

Cor In particular, $\Theta\mathcal{C}$ is a Reedy cat.

------, Θ_n ---------- $\square$

$\Theta_n^- \ni \alpha : \theta \longrightarrow \theta' \quad \Longleftrightarrow \quad T_{\theta'}$ is a <u>subtree</u> of T_θ.



// potential paper topic ... quasi-categories??

$\boxed{3/3}$

Lecture 12 $\qquad \underset{\wedge}{\times} \qquad$ 03/01/12

def $E_n\text{-}Sp :=$ full subcat of $sPSh(\Theta_n)$ st. $X(\theta_k) \sim *$ $k < n$

thm: [Boardman-Vogt, May, Segal] <u>vague form</u>

Every n-fold loop space $\Omega^n Y$ is canonically an E_n-space,

& conversely every E_n-space X is weakly equiv to $\Omega^n Y$ for some well-defined Y.

rmk: traditional formulation of this involves operad E_d^{top} of little discs in $D_d \subseteq \mathbb{R}^d$

We will relate $E_n\text{-}Sp$ to E_n^{top} & "factorization algebras" shortly

def functors

$$|\cdot| : E_n\text{-}Sp \rightleftarrows Sp_* : \Omega^n_E$$

Sp_* = pointed spaces (= pointed simplicial sets)

$i_{n-1} : \Theta_{n-1} \hookrightarrow \Theta_n$

by $|X| = \left(|X|_{\Theta_n}, \underbrace{|i_{n-1}^* X|_{\Theta_{n-1}}}_{} \right)$ contractible by def of $E_n\text{-}Sp$

$$\Omega^n_E (X, *) \overset{def}{=} \left[\theta \longmapsto \underset{Sp_*}{Map}(\left(|F_\theta|, |F_{\partial\theta}|\right), (X, *)) \right]$$

where $Sp((X,A),(Y,B)) = \{\varphi \in Sp(X,Y) \mid \varphi A \subseteq B\}$

$\boxed{1/5}$

$\partial\Theta = (i_{n-1})_!\,(i_{n-1})^* F_\Theta$ // can be defined // by Reedy structure // $\rightsquigarrow$ representable $\mathrm{Maps}_{Sp_*}((S^n,*),(Y,*))$

example ∂O_n as before. $\Omega^n Y$

$$\Omega^n_E Y(O_n) = \mathrm{Map}_{Sp}\big((|F_{O_n}|, |F_{\partial O_n}|), (Y,*)\big)$$
D_n, ∂D_n

lemma (i) ~~These are adjoint functors~~
(ii) Image of $|\cdot|$ has: $\pi_i(|X|) = 0$, $i<n$
(iii) if $Y \to Y'$ in Sp_* s.t. $\Omega^n_E Y \to \Omega^n_E Y'$ is a weak equiv,
& if Y & Y' fibrant, then $\pi_k(Y) \xrightarrow{\sim} \pi_k(Y')$ iso, $\forall k>n$
(iv) image of Ω^n_E has $\pi_0((\Omega^n_E Y)(O_n)) = \pi_n Y$ is a group.

So refine this (using lemma) to adjoint functors

$$|\cdot| : \begin{matrix}\text{group like}\\ E_n\text{-spaces}\\ (\text{i.e. } \pi_0 \text{ is gp})\end{matrix} \rightleftarrows \begin{matrix}\underline{n}\text{-connected}\\ \text{spaces}\end{matrix} : \Omega^n_E$$

Want this to be an equiv of model cats.
How do we make things into model cats?

- add a disjoint base pt : $Sp \underset{\text{forget}}{\overset{X \mapsto X \sqcup \{*\}}{\rightleftarrows}} Sp_*$
  use this to put a model cat on $\nearrow$ from struct on Sp.
- make n-connected spaces into a model cat via
  $\Sigma^n = S^n \wedge(\cdot) : Sp_* \rightleftarrows n\text{-Connected } Sp : \Omega^n$
  adjoint functors, $\nearrow$ a cofib generated model cat, with ~~gen~~ generating cofibs
  $\partial\Delta^n \hookrightarrow \Delta^n$, $n \geq 0$
  gen acyclic fib $\Lambda^n_k \hookrightarrow \Delta^n$
  and Ω^n preserves sequential colimits (as $(S^n,*) = (\Delta^n, \partial\Delta^n)$ is small ("compact" "perfect"))

or, much the same, you can Bousfield colocalize at objects $S^n, S^{n+1}, \ldots$

(Book: Frank Adams)
Similarly, can consider gp like E_n-spaces as a subcat, to add new weak equivs

thm (i) We have Quillen equiv

$$|\cdot| : (E_n\text{-}sp,\ \underset{S_n \nwarrow \text{Segal}}{\text{gp like}}) \rightleftarrows n\text{-connected spaces} : \Omega^n_E$$

(ii) the map $(E_n\text{-}Sp, \underset{S_n}{\text{gp like}}) \longrightarrow Sp$ $\quad X \mapsto \overline{X}(O_n)$ is homotopy conservative
i.e., if $\varphi : X \to X'$ has $\overline{X}(O_n) \to \overline{X}'(O_n)$
is a weak equiv, then φ is a weak equiv.

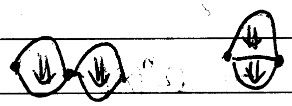
(this is a precise form of the "vague" thm)

Observe (i) says, in particular, $X \to \Omega^n_E|X|$ is a weak equiv (in $(E_n\text{-}Sp, \underset{S_n}{\text{gp like}})$)
so every E_n-space is a loop space.

(ii) though it requires extra data to recover Y from $\Omega^n Y$

(i.e. the data of being an E_n-space, i.e. roughly

n commuting (up to homotopy) multiplications, which are assoc up to homotopy



$\Omega^n Y \to \Omega^n Y'$

a map $\varphi : X \to X'$ which induces a weak equiv $\overline{X}(\mathcal{O}_n) \to \overline{X}'(\mathcal{O}_n)$

$Y \to Y'$ $Y = |X|$, $Y' = |X'|$ is already a weak equiv

Pf of lemma: (i) clear adjoint, as both defined by the

colt $\Theta_n \to$ n-sphere $\quad \theta \mapsto (|F_\theta|, |F_{\partial\theta}|) \in Sp_*$

note if $\overline{X}(\mathcal{O}_a) \sim *$, $a < n$, then $|i^*_{n-1} X| \sim *$ also,

so $\theta \mapsto$ wedge of spheres, # of spheres = # of leaves of ht n in T_θ

(ii) $|X|$ is a colimit of S^k, $k \geq n$, & S^i is small/compact, so commutes with countable colimit, & $\pi_i(S^k) = 0$ if $i < n \leq k$.

(iii) $Y \to Y'$ a map has $\Omega^n Y \to \Omega^n Y'$ w.e. if

$$\pi_i(\Omega^n Y) \xrightarrow{\sim} \pi_i(\Omega^n Y') \text{ iso}, \ \forall i \geq 0$$

$$\pi_{i+n}(Y) \to \pi_{i+n} Y'$$

But then (***) Y fibrant $\Rightarrow \Omega^n_E Y$ Segal fibrant, (pf: exercise, properly of $\mathrm{Map}_{sp}(\ ,\ )$)

i.e. $\Omega^n_E(Y)(\theta) \sim (\Omega^n Y)^{\wedge k}$, $k = \#$ of leaves of top ht in T_θ

so a weak equiv $\Omega^n Y \to \Omega^n Y'$ induces a weak equiv for all pasting diagrams $\theta \in \Theta_n$, i.e. a levelwise w.e. of $sPSh(\Theta_n)$

(iv) "obvious". $\square$

Pf of thm: non-formal issue is to prove $\overline{X}(\mathcal{O}_n) \to \Omega^n |X|$ is a w.e., if X is fib & cofib.

Factor this map, using

$$\begin{array}{ccc} [1](\theta) & \Theta_n & \xleftarrow{S_{\Theta_{n-1}}} \Delta \times \Theta_{n-1} \\ \uparrow & \uparrow \sigma_{n-1} & \nearrow ([1],\theta) \\ \theta & \Theta_{n-1} & \theta \mapsto \end{array}$$

induces

$$\begin{array}{ccc} E_n\text{-}Sp & \longrightarrow & sPSh(\Delta \times \Theta_{n-1}) \\ \sigma^*_{n-1} \downarrow j & & \| \\ & & \Delta^{op}(sPSh(\Theta_{n-1})) \\ & & \cup \\ E_{n-1}\text{-}Sp & \xleftarrow{u} & \Delta^{op}(E_{n-1}\text{-}Sp) \end{array}$$

$$Y([1](\cdot)) \leftarrow\!\shortmid Y$$

U-GRADE

$X(\Theta_n) = (\sigma_{n-1}^* X)(\Theta_{n-1}) \xrightarrow{\sim} \Omega^{n-1} |\sigma_{n-1}^* X|$ $\qquad |\delta^*_{\Theta_{n-1}} X|$

is a w.e., by induction, as we've already seen X Segal fibrant $\Rightarrow \sigma_{n-1}^* X$ Segal fibrant, & similarly for gp like condition

$= \Omega^{n-1} u |\delta^*_{\Theta_{n-1}} X|_{\Theta_{n-1}}$ $\qquad$ [Consider $u : \Delta^{op}Sp \to Sp$, $(\cdot)([1])$]

$\xrightarrow{\sim} \Omega^{n-1} \Omega |X|_{\Theta_n} = \Omega^n |X|$

X Segal fibrant + gp like $\Rightarrow \delta^*_{\Theta_{n-1}} X$ is

So it remains to prove $n=1$: [In paper of Segal]

$X_\bullet \in \Delta^{op} Sp$ s.t. $X_0 \sim *$ contractible

$X_n \to X_1^n$ is a weak equiv, & fibration

& X_1 fibrant, $\pi_0 X_1$ is a gp

WANT: $X_1 \to \Omega |X|$ is a weak equiv, ie.

$\pi_i(X_1) \to \pi_{i+1}(|X_\bullet|)$ is a w.e.

$$Y = |X_\bullet| \longleftrightarrow \Big| \big(\cdot \leftleftarrows \underset{\Omega Y}{\overset{\shortparallel}{X_1}} \leftleftarrows\!\!\!\leftarrow \underset{(\Omega Y)^2}{X_1^2} \quad \cdots \big) \Big|$$

"is" the Bar complex for the "gp" ΩY, ie "is $B\Omega Y$"

if G a gp $(1 \leftleftarrows G \leftleftarrows\!\!\!\leftarrow G^2 \cdots) = N(\quad)$

note $\Omega|X|$ is homotopy fiber product $\quad \begin{array}{ccc} \Omega|X| & \to & * \\ \downarrow & \square_h & \downarrow \\ * & \to & |X| \end{array}$ $\quad$ So

ets:

$$\begin{array}{ccc} X_1 & \to & |PX| \\ \downarrow & & \downarrow \\ X_0 & \to & |X| \end{array} \quad \text{cartesian}$$

& PX path space of X, $(PX)_n = X_{n+1}$, $(PX)(\alpha) = X(\alpha')$

$\alpha'[0] = [0]$

$\alpha'[i] = \alpha([i-1]) + 1$, $i > 0$.

Contractible via std homotopy ~~PX~~

$PX \times \Delta^1 \to PX$.

Can show directly that this cartesian $\Leftrightarrow \pi_0 X_1$ is a gp.
(note $X_n \sim X_1^n$)

Slightly bogus pf: there is a convergent s.s. $E_2^{pq} = \pi_p^h \pi_q^v (X_\bullet) \Rightarrow \pi_{p+q}(|X|)$
if $q \geq 1$

& as $\pi_0 X_1$ is a gp, always, is ok if $q=0$ also.

So $\pi_q^v(X_\bullet) = (0 \leftleftarrows \pi_q X_1 \leftleftarrows\!\!\!\leftarrow (\pi_q X_1)^2 \cdots) = B(\pi_q X_1)$ precisely

So $\pi_i(B(\pi_q X_1)) = \begin{cases} \pi_q X_1 & \text{if } i=1 \text{ as } \pi_q X_1 \text{ discrete} \\ 0 & \text{otherwise} \end{cases}$

So s.s. collapses to give $\pi_{q+1}|X| \longleftarrow \pi_q X_1$ iso, $\forall q \geq 0$ as desired, directly.

Lecture 13

03/06/12

$| \ | : (E_n\text{-}Sp, \text{Segal}, \text{gp-like}) \rightleftarrows$ n-connected $Sp_* : \Omega^n_E$

$X \mapsto \tilde{X}(\mathcal{O}_n)$

maybe better name is B^n:

write $\Theta_n\text{-}Sp^{Rezk} := (sPSh(\Theta_n), \text{Segal}_a, \text{Completeness}_a, \forall a \leq n)$

E_d-monoidal $\Theta_n\text{-}Sp^{Rezk} := \left(sPSh(\Theta_{n+d}), \begin{array}{ll} \text{Segal}_a & \forall a \leq n+d \\ \text{Completeness}_a, & n+d \geq a \geq d \\ \tilde{X}(\mathcal{O}_a) \sim *, & a < d \end{array}\right)$

So $n=0$ is E_d-Sp; ~~[illegible]~~

expect: Quillen adjoint functors

$B^d : E_d$-monoidal $\Theta_n\text{-}Sp^{Rezk} \rightleftarrows \Theta_n\text{-}Sp_*^{Rezk} : \Omega^d_E$

$\downarrow U = (\sigma^d)^*$

$\Theta_n\text{-}Sp^{Rezk}$ $\quad U(X)(\theta) = X(\sigma^d(\theta))$, $\sigma\theta = [1](\theta)$ suspension

s.t. (essential) • image of B^d = "d connected Θ_n-spaces"

• image of Ω^d_E = "gp like E_d-monoidal Θ_n-spaces"

• U is homotopy conservative $\quad // \ U\Omega^d_E = \Omega^d$

Moreover, when restricted to these image subcats, B^d, Ω^d_E give Quillen equivs

The case $n=0$ is the theorem above.

Moreover, there are Quillen adjoint functors.

$\Sigma^d : \Theta_n\text{-}Sp_*^{Rezk} \rightleftarrows \Theta_n Sp_*^{Rezk} : \Omega^d$

with images: $Im\,\Omega^d$ = gp like Θ_n-spaces

$Im\,\Sigma^d$ = d-connected Θ_n-Sp.

In fact $B^d : E_{d+d'}$-monoidal $\Theta_n\text{-}Sp_*^{Rezk} \rightleftarrows E_{d'}$-monoidal $\Theta_n Sp_*^{Rezk} : \Omega^d_E$

$\downarrow U_d = (\sigma^d)^*$

$E_{d'}$-monoidal Θ_n-Sp.

Moreover, $U\Omega^d_E X = \Omega^d X$ as before.

Rmk: all of these model cats $(sPSh(\Theta_n), W)$ ← some localizing sets

are supposed to model kinds of "weak n cats". More precisely, can regard $(sPSh(\Theta_n), W) \in (sPSh(\Theta_{n+1}), Rezk)$ // using simplicial nerve

Want to say (& should say!) these Quillen functors $F : ( ) \rightleftarrows ( ) : G$ give rise to honest morphisms in $\Theta_{n+1}Sp$ $[F] : ( ) \rightleftarrows ( ) : [G]$ which are w.e. if Quillen equiv (should check this!)

Spectra, vaguely

let $\text{Spectra} = \varprojlim_{\Omega} Sp_* = (\cdots \xrightarrow{\Omega} Sp_* \xrightarrow{\Omega} Sp_*)$ (*)

explicitly, an object is a sequence $X_d \in Sp_*$ with morphisms

$\alpha_d : X_d \to \Omega X_{d+1}$

(or equiv, by adjointness, $\Sigma X_d \to X_{d+1}$)

// Ω means maps_* from S^1

We have enough technology to interpret (*) scientifically

(it's in a diagram cat / cofibred over $\mathbb{N}^{op}$)

& can take limits as CSS ...

or as some model cat, Here it is, by hand:

Model cat str on Spectra: [Bousfield-Friedlander]

$f: X \to Y$ fibrant if $\begin{cases} f_d : X_d \to Y_d \text{ fibrant in } Sp, \ \& \\ X_d \to \Omega X_{d+1} \times_{\Omega Y_{d+1}} Y_d \text{ is a w.e. in } Sp \end{cases}$

In particular, X fibrant $\Leftrightarrow$

each X_d is fibrant, and $X_d \to \Omega X_{d+1}$ is w.e.

so $X_0 \xrightarrow{\sim} \Omega X_1 \xrightarrow{\sim} \Omega^2 X_2 \xrightarrow{\sim} \cdots$

each X_d admits an ∞ # of deloopings "is an ∞-loop space"

If $X \in$ Spectra, define $X^f \in$ Spectra by $(X^f)_d = \varinjlim \Omega^k X_{d+k} = \lim(X_d \to \Omega X_{d+1} \to \Omega^2 X_{d+2} \to \cdots)$

Obvious that $(X^f)_d \to \Omega(X^f)_{d+1}$ is a weak equiv;

so X^f is fibrant, & $X \to X^f$

Now declare $X \to X^f$ to be a w.e., & so a fibrant replacement.

more generally, $X \to Y$ is a w.e. if $X^f \to Y^f$ is a levelwise w.e. in Sp.

What have we done?

Note $\pi_i (X^f)_d = \varinjlim \pi_{i+k}(X_{d+k})$

Now, if $X_d = \Sigma^d \bar{X}$; $\bar{X} \in Sp$, then $\varinjlim \pi_{i+k}(X_{d+k}) = \varinjlim \pi_{i+k}(\Sigma^k \Sigma^d \bar{X})$

Recall if $|X|$ is a finite CW complex; (ie $X \in Sp$ has only finitely many non-degen cells)

then $\pi_{i+k}(\Sigma^k X) = \pi_{i+k+1}(\Sigma^{k+1} X) =: \pi_i^s(X)$ for $k \gg 0$

are called the stable homotopy gps of X.

"Whitehead's thm"

Thm (BF) With these fibrations & w.e., Spectra is a cofib generated model cat,

Quillen adjoint: $\Sigma^\infty : Sp_* \rightleftarrows \text{Spectra} : \Omega^\infty$ $(\Omega^\infty Y) = Y_0$

$\Sigma^\infty X = (X, \Sigma X, \Sigma^2 X, \ldots)$, map $\Sigma(\Sigma^\infty X)_n \to (\Sigma^\infty X)_{n+1}$ is $\Sigma\Sigma^n X \xrightarrow{\simeq} \Sigma^{n+1} X$.

s.t. if $X \in$ Spectra, define ΣX by $(\Sigma X)_i := \Sigma(X_i)$

ΩX by $(\Omega X)_i = \Omega(X_i)$

and $\Sigma : \text{Spectra} \rightleftarrows \text{Spectra} : \Omega$ are adjoint

note that $\Sigma \Sigma^\infty X = \Sigma^\infty \Sigma X$, but $\Sigma^\infty(\Omega X) \neq \Omega(\Sigma^\infty X)$. Instead

Lem $X \in$ Spectra $\Rightarrow$ $X \leftrightarrow \Omega\Sigma X$, $\Sigma\Omega X \to X$ weak equivs

• is a universal functor; for any presentable pointed cat $\mathcal{C}$
Spectra

$\mathcal{C} \rightsquigarrow \text{Spectra}(\mathcal{C})$

st. $\text{Spectra}(\mathcal{C})$ satisfies universal property "stable"

$X \in Sp$, $X \to \Omega^\infty \Sigma^\infty X = \varinjlim \Omega^i \Sigma^i X$

replaces X with an ∞-loop space, $\pi_i(\Omega^\infty \Sigma^\infty X) = \varinjlim_k \pi_{i+k}(\Sigma^k X) = \pi_i^s(X)$

// Aside: Goodwillie Calculus

exercise

note $\pi_{i+k}(\Sigma^k X) = 0$ if $i<0$, as $\Sigma^k X$ built out of spheres $S^{k'}$, $k' \geq k$

and, if $Y \in$ Spectra, $\Sigma^\infty \Omega^\infty Y \to Y$ induces isos of π_i's $i \geq 0$,

& $\pi_i(\Sigma^\infty \Omega^\infty Y) = 0$, $i<0$, unlike $\pi_i(Y)$.

So image of Σ^∞ is 0-connected Spectra, image of Ω^∞

is infinite loop spaces in Sp.

& we can write $d=\infty$ version of Segal-May-BV theorem

this is due to Segal, 1974

thm: Quillen-equiv:

$(\varinjlim E_d\text{-}Sp, \text{Segal gp like}) \rightleftarrows \text{0-connected Spectra} : \Omega^\infty$

$\uparrow$

$(sPSh(\varinjlim \Theta_d), \text{Segal gp like}) \xrightarrow{U} Sp$

// I don't think this is true...

$\boxed{\begin{array}{l}\text{Prop: } \varinjlim \Theta_d = (\Theta_0 \to \Theta_1 \to \dots) \overset{??}{\simeq} (\text{FinSet}_*)^{op} =: \Gamma \quad \text{"Segal's cat"} \\ \text{\& a } \Gamma\text{-space is a space with a homotopy assoc \& commutative operation} \\ \qquad +: X \times X \to X\end{array}}$

$\varinjlim E_d\text{-}Sp$ = "E_∞-Sp", which is also called a Γ-space.

Lecture 14 Little disks Operad

03/08/12

$E_d(n)$ = space of n disjoint open d-dim cubes in standard open cube $(0,1)^d \subseteq \mathbb{R}^d$

cube $= (a_1, b_1) \times \dots \times (a_d, b_d)$ $a_i < b_i$

homeo to space of n disjoint cubes in $\mathbb{R}^d$

& $\mathbb{R}^d \times \mathbb{R}^*$ acts on by rescaling and translation

U-GRADE

$E_d(n) \longrightarrow \mathrm{Conf}_n(0,1)^d$ $\qquad$ $\mathrm{Conf}_n X = \{(x_1,\dots,x_n) \in X \mid x_i \neq x_j\}$

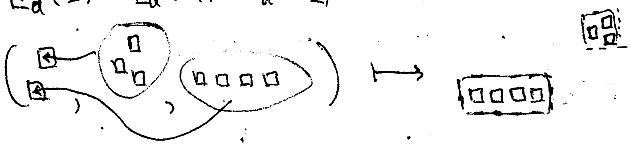
$\text{cube}_1,\dots,\text{cube}_n \longmapsto \text{center of cube}_1,\dots,\text{cube}_n$

this map is homotopy equiv

map $E_d(n) \times E_d(k_1) \times \dots \times E_d(k_n) \longrightarrow E_d(k_1 + \dots + k_n)$

use translation & rescale, take each k_i-tuple of cubes & place it in ith cube in $E_d(n)$

$E_d(2) \times E_d(k_1) \times E_d(k_2)$

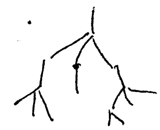


Structure makes E_d into an operad on Top spaces

If $\mathcal{C}$ is a closed sym. monoidal cat (eg. Top, --)

$Y \in \mathcal{C}$, then operad: $\mathrm{CoEnd}(Y)$ (comult category)

$$n \longmapsto \mathcal{C}(Y, Y \otimes \dots \otimes Y) = \mathcal{C}(Y, Y^{\otimes n})$$



& operad structure is defined by composition

this acts, for any $X \in \mathcal{C}$, on $\mathcal{C}(Y, X)$

i.e. $\mathcal{C}(Y, Y^{\otimes n}) \otimes \mathcal{C}(Y,X)^{\otimes n} \longrightarrow \mathcal{C}(Y,X)$

example: Take $\mathcal{C} = \mathrm{Top}_*$, $\otimes = \wedge$, $Y = (S^n, *)$

$$\mathrm{Top}_*(S^n, X) =: \Omega^n X$$

this is an algebra for the operad $\mathrm{CoEnd}(S^n)$, hence an algebra for the suboperad E_d

i.e. $\Omega^n X$ is an algebra for the operad E_d

thm [BV, May, Segal]

$$B^d : \left(\begin{array}{c}\text{gp like } E_d \text{ algebras} \\ \text{in } \mathrm{Top}_*\end{array}\right) \rightleftarrows \begin{array}{c}d\text{-connected} \\ \text{spaces}\end{array} : \Omega^d$$

$$\Big\downarrow\, U = \text{underlying set}$$

$$\text{gp like } \mathrm{Top}_*$$

B^d, Ω^d are equiv of Quillen model cats, U homotopy conservative

This screams the following:

thm: equiv of model categories E_d-monoidal Θ_n-Sp $\rightleftarrows$ E_d-Alg(Θ_n-Sp).

examples

$d=0$: $E_d(0) = *$, $E_d(n) = \phi$, $n > 0$

$d=1$: $E_1(n) = n$ disjoint intervals $I_1, \dots, I_n$ in $\mathbb{R}$

$\xrightarrow{\text{homotopic}} S_n$ $\qquad$ $\exists!\, \sigma \in S_n$ s.t. $I_{\sigma_1} < I_{\sigma_2} < \dots < I_{\sigma_n}$.

$I_\bullet \rightsquigarrow \sigma_\bullet$

thm is more or less a ~~tautology~~ (= def of RHS) for $d=1$

Issue is $d>1$

for $d>1$, have

thm: "Dunn's thm" $\underset{\text{operad}}{E_{d+d'}} \simeq E_d \underset{\text{tensor product}}{\otimes} E_{d'}$, equivalently (✱✱)

$$E_d\text{-Alg}\,(E_{d'}\text{-Alg}(\mathcal{C})) \simeq E_{d+d'}\text{-Alg}(\mathcal{C}).$$

For the LHS, we essentially know this already, & so Dunn's thm + $d=1 \Rightarrow$ thm.

So from this optic, point of thm is (✱✱), is an analysis of the homotopy type of little disks operad.

We're going to sketch a direct pf, or rather some ingredients of a direct pf.

(the explicit combinatorics we use also appear in Dunn's thm).

WANT to explicitly study homotopy type of $E_d(n)$ $\forall n$.

& in particular, find a poset A s.t. $|NA| \sim E_d(n)$

—— We will do this "dumbly" ——

Let $(A,\leq)$ poset, $X \in \mathrm{Top}$, $\forall \alpha \in A$, $C_\alpha \subseteq X$ contractible subspace of X

st. (i) $C_\alpha \subseteq C_\beta \iff \alpha \leq \beta$ (ii) $C_\alpha \hookrightarrow X$ closed embedding, ie. a cofibration

(iii) "$\bigcup C_\alpha = X$", $\underset{\text{in particular}}{\longleftarrow}$ $\lim\limits_{\overrightarrow{A}} C_\alpha = X$ "cellular decomp of X"

then $|NA| \underset{\text{each cell is contractible}}{\xrightarrow{\sim}} \operatorname{holim}\limits_{\overrightarrow{A}} C_\bullet \underset{\text{as each inclusion is a cofibration}}{\xleftarrow{\sim}} \operatorname{colim}\limits_{\overrightarrow{A}} C_\bullet = X$ (✱✱✱)

For $d=2$, Fox–Neuwirth for $E_2(n)$ found a cell decomposition

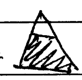
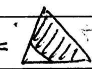
$$\left[\begin{array}{l} E_d(1) \sim * \\ E_d(2) \sim S^{d-1} \end{array}\right]$$

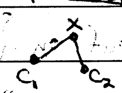
& now for $d>2$, exists an analog of Fox–Neuwirth decomp,

first written by Getzler–Jones, 1994

but it isn't a cell decomp.... (wrong; fixed)

Instead: define $\mathring{C}_\alpha = C_\alpha \setminus \bigcup\limits_{\beta<\alpha} C_\beta$

example: $X =$  $C_1 =$  $\mathring{C}_2 =$  $C_0 = X$

$\mathring{C}_0 \neq \emptyset$ 

def: C_α is "redundant" if $\mathring{C}_\alpha = \emptyset$, non-redundant otherwise

lemma: if (i)' $\alpha \leq \beta \Rightarrow C_\alpha \subseteq C_\beta$ & if $\mathring{C}_\alpha \neq \emptyset$, then $C_\alpha \subseteq C_\beta \Rightarrow \alpha \leq \beta$

and (ii), (iii), then $|NA| = X$.

U-GRADE

pf Let A' be subposet of non-redundant cells, then still case that
$\varinjlim_{A'} C = X$, so $X = |NA'|$

by (**). Now inclusion $|NA'| \to |NA|$: homotopy fibers of this map are $|NF_\alpha|$, where $\alpha \in A$, $F_\alpha = \{\beta \in A' \mid \beta \le \alpha\}$

by (**) $|NF_\alpha| = \varinjlim_{\beta \in F_\alpha} C_\cdot = C_\alpha$ (as either $\alpha \in A'$, in which case it's clear
or $\alpha \notin A'$, in which case it's the def of redundant)

Now "Quillen thm A" $\Rightarrow$ $|NA'| \to |NA|$ w.e.

Prop: (Fiedorowicz, Berger) $E_d(n)$ admits such a generalized cell structure

Following variant of the lemma is more useful

lemma: Let A be a Reedy cat, $C: A \to \{\text{subspaces of a top space } X\}$
st. (ii) natural map Latching object.
$L_a(C) \to C(a)$ is a cofib
$\quad \| $
$\varinjlim_{b \to a,\, \deg(b) < \deg(a)} C(b)$

(iii) $\varinjlim_{A} C \to X$ is a w.e.

(o) $C(a)$ contractible $\forall a$

Then $|NA| \xrightarrow{\sim} X$.

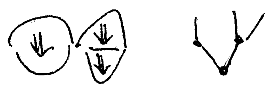
the issue is redundant cells: you can't just throw them away b/c higher dim ones need redundant lower dim guys to glue properly
desired: e.g.?

$x, y \in \mathbb{R}^d \quad x = (x_1, \dots, x_d)$

$x \le^{lex}_i y$ if $x_a = y_a$, $a < i$, $x_i < y_i$

ob $\Theta^{fr}_{d,A} = \{(\theta, \alpha) \mid \theta \in \Theta_d$, α: bijection between leaves of T_θ & $A\}$

A finite set



thm: $|N\Theta^{fr}_{d,\{1,\dots,n\}}| \xrightarrow{\sim} \mathrm{Config}_n(\mathbb{R}^d)$

$E_d(n) \times E_d(k_1) \times \dots \times E_d(k_n) \longrightarrow E_d(k_1 + \dots + k_n)$

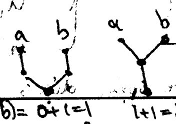
$\Theta^{fr}_{d,[n]} \times \Theta^{fr}_{d,[k_1]} \times \dots \times \dots \longrightarrow \dots$

"Ran space"

repackage E_d-Alg into "factorization alg" (cosheaf on $\{S \subset \mathbb{R}^d \mid \#S \text{ finite}\}$
$\qquad \| $
$\qquad \bigcup_{k \ge 0} S^k \mathbb{R}^d$

Lecture 15 03/13/12

$|N\Theta^{fr}_{d,\{1,\dots,n\}}| \xrightarrow{\simeq} E_d(n) \simeq \mathrm{Config}_n(\mathbb{R}^d)$



If $(\Theta,\alpha) \in \Theta^{fr}_{d,A}$ this defines a partial order on A

by $a < b$ iff the leaf labelled by a is to the left of the leaf labelled by b

define $C(\Theta,\alpha) = \{(z_1,\dots,z_n) \in \mathrm{Conf}_n(\mathbb{R}^d) \mid \text{if } a < b, \text{ then } z_a <^{lex}_{b(a,b)} z_b\}$

where $b(a,b) = 1 + (\text{level of the tree } T_\Theta \text{ at which point leaves labelled by } a \,\&\, b \text{ meet})$

(i) $C(\Theta,\alpha)$ is the interior of a convex set in $\mathbb{R}^{dn}$

(ii) If restrict to $\Theta'^{fr}_{d,\{1,\dots,n\}} = \{(\Theta,\alpha) \mid T_\Theta \text{ only leaves at level } n\}$ $\Theta = [0]$

then $\lim\limits_{\Theta'^{fr}_{d,\{1,\dots,n\}}} C(\Theta,\alpha) = \mathrm{Conf}_n(\mathbb{R}^d)$

Given $(z_1,\dots,z_n)$, $i<j$, set $b(i,j) = \max\{k \mid (z_i)_a = (z_j)_a, a \leq k\}$ $\{0,\dots,d\}$

[just "lexicographically" order z_i's]

$\exists!$ tree T with all leaves at top level & branching points $b(i,j)$ so $T = T_\Theta$

If $(\Theta',\alpha') \in \Theta'^{fr}_{d,\{1,\dots,n\}}$ & $z \in C(\Theta',\alpha')$ also $z \in C(\Theta,\alpha)$

$\Leftrightarrow$ $\exists$ morphism $(\Theta',\alpha') \to (\Theta,\alpha)$

(iii) $L_a(C) \to C(a)$ is the inclusion of a face of a convex polyt. into the convex poly, & so a closed embed. (cofib)

so $|N\Theta'^{fr}_{d,\{1,\dots,n\}}| \longrightarrow \mathrm{Config}_n(\mathbb{R}^d)$

$\downarrow$ w.e.

$|N\Theta^{fr}_{d,\{1,\dots,n\}}|$

don't even need to check condition (iii) for all of $\Theta^{fr}_{d,A}$, etc

$i: \Theta'^{fr}_d \hookrightarrow \Theta^{fr}_d$ has a right adj r

$r(\Theta) = \Theta'$, if $T_{\Theta'}$ is T_Θ with leaves not at top level pruned.

think this ~~follows from Reedy stuff~~

$E_d(n) \times E_d(k_1) \times \dots \times E_d(k_n) \longrightarrow E_d(k_1 + \dots + k_n)$

poset $X_d(n)$

st. $|X_d(n)| \sim E_d(n)$ $\rightsquigarrow$ construct an operad in posets out of this

VARIATIONS: $\mathcal{C}$ be a sym. monoidal (∞-) cat

X manifold, $\dim X = d$, $E_X(n) =$ disjoint open discs in X

$E_X(n) \times E_d(k_1) \times \dots \times E_d(k_1 + \dots + k_n)$ "operad" (sort of)

so $E_{\mathbb{R}^d}(\cdot) =$ little disc operad, $E_X(1) \to X$ w.e. $E_X(0) = *$

U-GRADE

prop: $E_X\text{-Alg}(\mathcal{C}) \simeq \left\{ A : \left\{\begin{array}{l}\text{cat of disjoint open discs} \\ U_1, \dots, U_n \text{ in } X \\ n \text{ varies}\end{array}\right\} \longrightarrow \mathcal{C} \right.$

+ data of

(i) if $V_1, \dots, V_n$ are open discs, all contained in U, an open disc

$$A(V_1) \otimes \cdots \otimes A(V_n) \longrightarrow A(U)$$

(ii) If $V \subseteq U$ inclusion of open disc $\subseteq$ open disc, this map $A(V) \longrightarrow A(U)$ equiv in $\mathcal{C}$. $\}$

def: $\text{Ran}(X) = \left\{ S \subseteq X \;\middle|\; \begin{array}{l}\# S < \infty \\ S \neq \emptyset\end{array} \right\}$ nonempty finite subset of X

topologized $\quad \text{Ran}\, X = \varinjlim \text{Ran}^{\leq n}(X)$

closed subsets $\quad \text{Ran}^{\leq n}(X) = \{ S \subseteq X \mid 0 < \# S \leq n \} \twoheadleftarrow S^n X$

$\text{Ran}^n(X) = \{ S \subseteq X \mid \# S = n \} = \text{Config}_n(X) \hookrightarrow \text{Ran}^{\leq n}$ open set

thm: $\text{Ran}\, X$ weakly contractible

Say a sheaf $\mathcal{F}$ on $\text{Ran}(X)$ is <u>constructible</u> if it is constr wrt the stratification, i.e.

(i) $\mathcal{F} = \varprojlim i_{n*} i_n^* \mathcal{F}$, $\quad i_n : \text{Ran}^{\leq n}_X \hookrightarrow \text{Ran}\, X$

(ii) $j_n^* i_n^* \mathcal{F}$ is a locally const. sheaf on $\text{Config}_n(X)$, $j_n : \text{Config}_n(X) \hookrightarrow \text{Ran}^{\leq n} X$.

prop:

$\mathcal{F}$ constr $\Leftrightarrow$ $\forall U_1, \dots, U_n$ disjoint open disc

$V_1, \dots, V_n$ discs st. $V_i \subseteq U_i$

$\mathcal{F}(\text{Ran}(\amalg U_i)) \longrightarrow \mathcal{F}(\text{Ran}(\amalg V_i))$ is a w.e.

+ condition equiv to (i) "hypercompleteness" $\leftarrow$ due to Lurie.

---

def: a cosheaf on $\text{Ran}\, X$ is a functor $\mathcal{F} : \left(\begin{array}{c}\text{cat of open} \\ \text{sets of } X\end{array}\right) \longrightarrow \mathcal{C}$ st. $\forall c \in \mathcal{C}$

$\mathcal{F}_c : U \longmapsto \mathcal{C}(\mathcal{F}U, c)$ is a sheaf on $\text{Ran}\, X$

it is constructible if $\mathcal{F}_c$ is.

def [BD]: a "factorizable cosheaf" is a constr cosheaf $\mathcal{F}$ on $\text{Ran}\, X$ st. $\forall U, V \subseteq \text{Ran}\, X$ independent, the map $\mathcal{F}U \otimes \mathcal{F}V \xrightarrow{\sim} \mathcal{F}(U * V)$ is an equiv in $\mathcal{C}$.

$U * V = \{ S \cup T \mid S \in U, T \in V \}$

Put $\text{Supp}(U) = \bigcup_{S \in U} S \subseteq X$. Say U, V <u>independent</u> if $\text{supp}\, U \cap \text{supp}\, V = \emptyset$

If U, V independent, $\quad U \times V \longrightarrow U * V \subseteq \text{Ran}\, X \quad$ is a homeo.

$(S, T) \longmapsto S \cup T$

prop $E_X\text{-Alg}(\mathcal{C}) \longrightarrow$ Factorizable Cosheaves on $\text{Ran}\, X$, is an equiv of $(\infty\text{-})$cat's

---

X alg variety, $\text{Ran}\, X$ ind-alg variety as well as top space.

X alg curve

G alg gp. ~~reductive~~ semisimple

Weil uniformization: if $\mathcal{G}$ is a princ G-bundle on X (étale locally)

X curve, $x \in X$, $\mathcal{G}|_{X-\{x\}} \simeq G \times (X-\{x\})$

$\hat{\mathcal{S}}$ formal disc around x. $\mathcal{G}|_{\hat{\mathcal{S}}} \simeq G \times \hat{\mathcal{S}}$ $\qquad \hat{\mathcal{S}} = \mathrm{Spf}\ \mathbb{C}[\![x]\!]$

so data of $\mathcal{G}$ is really $\varphi : \hat{\mathcal{S}} \setminus \{x\} \longrightarrow G$ $\qquad \varphi \in G((x))$

"Spec $\mathbb{C}((x))$" $\qquad$ " $\shortparallel$ maps(S^1, G)"

$$\mathrm{Bun}_G = G_{out} \backslash G((x)) / G[\![x]\!] \qquad G[\![x]\!] = \mathrm{maps}(\hat{\mathcal{S}}, G)$$

You can make sense of this in alg. geom. $\qquad G_{out} = \mathrm{maps}(X - x, G)$

Set $\mathrm{Gr}_x = G((x))/G[\![x]\!]$ this is an honest Ind alg variety, a direct limit of f.d. proper varieties

"loop Grassmanian"

$G((x)) \sim \mathrm{maps}(S^1, G)$

$G[\![x]\!] \sim \mathrm{maps}(\mathrm{Disk}, G) \sim G$

So $\mathrm{Gr}_x \sim \Omega G = \mathrm{maps}((S^1, *), (G, 1))$

So Weil uniformization gives $G_{out} \backslash \Omega G \xrightarrow{\sim} \mathrm{Bun}_G$

ΩG is $\Omega^2 BG$, i.e. $\underset{\sim}{\mathrm{Gr}}$ is a double loop space,

so E_2-Algebra, so gives factorizable cosheaf on $\mathbb{R}^2$

BD-Grassmanian is:

$S \subseteq X \rightsquigarrow \mathrm{Gr}(S) = \prod_{x_i \in S} \mathrm{Gr}_{x_i}$ gives a factorizable Ind-Alg variety $\underset{\sim}{\mathrm{Gr}}$

$$\begin{array}{ll} \underset{\sim}{\mathrm{Gr}}|_{\mathrm{Ran}^{\leq n} X} & \leftarrow \text{ind alg. variety} \\ \downarrow & \text{reasonable map of alg variety} \\ \mathrm{Ran}^{\leq n}_X & = \text{alg variety} \end{array}$$

Thm: this map is flat! // ∞-dim'l needed, otherwise f.d. fibers have dim go down!

// Any kind of objects on $\underset{\sim}{\mathrm{Gr}}$ give us factorizable cosheaf in those objects

// This is what Vertex alg is; BD $\rightsquigarrow$ Chiral algebras.

// Sheaves on $\underset{\sim}{\mathrm{Gr}}$ are "Hecke operators" for Langlands.

The homotopy theory of curves for ∞-dim'l $\leftarrow$ input is what we've been doing.

— FIN —