PART III SEMINAR (LENT TERM): MODULI STACKS OF VECTOR BUNDLES

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1. Introduction

Let X be a smooth projective curve over $\mathbf{C}$ (you can think of X as a Riemann surface in the analytic setting if you wish) throughout. Then there exists a smooth variety Pic_X over $\mathbf{C}$ whose $\mathbf{C}$-points are the isomorphism classes of line bundles on X. More generally, for any scheme (variety) S over $\mathbf{C}$, the set $\mathrm{Hom}_{\mathbf{C}}(S, \mathrm{Pic}_X)$ corresponds to a sheafification of the line bundles on the family of curves $X \times S$ (the Picard group). This is an example of a moduli problem: studying a class of objects (line bundles) on families of schemes (curves). In this case, the Picard scheme Pic_X is the fine moduli space of line bundles.

We will be considering the moduli problem of vector bundles of a fixed rank r on $X \times S$ as S varies. Pullback makes this a "presheaf" on the category of **Sch**. But what category does this functor go to? Vector bundles have many non-trivial automorphisms ($\mathrm{Aut}(\mathcal{O}^r) = \mathrm{GL}_r$), so we shouldn't just ignore these. To account for this, we define a presheaf $\mathrm{Bun}_r : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Gpd}$ to the 2-category of groupoids. Since you can glue compatible vector bundles together, this presheaf is a sheaf/stack. The goal of this talk will be to outline/convince you that this sheaf has geometric properties which makes it behave almost like a scheme, i.e., Bun_r is an algebraic stack [Beh91, Bro10, Ols06, Sor00].

This result is important because once we know Bun_r is algebraic, one can define quasi-coherent sheaves and $\mathcal{D}$-modules on it, just as with schemes. The geometric Langlands program studies the correspondence between $\mathcal{D}$-modules on Bun_r ($\leftrightarrow$ automorphic forms) and local systems (rank r vector bundles with flat connections) on X ($\leftrightarrow$ Galois representations), and this correspondence has deep connections to both number theory (classical Langlands) and quantum physics (cf. [Fre10]).

2. Algebraic stacks

For $S \in \mathbf{Sch}$, let $\mathrm{Bun}_r(S)$ be the groupoid consisting of vector bundles $\mathcal{E}$ of rank r on $X_S := X \times S$, where morphisms are isomorphisms of vector bundles over X_S. Remark: vector bundles (as schemes over X) are equivalent to locally free sheaves on X. If we have a map $f : T \to S$, then the pullback $f^*\mathcal{E}$ is a vector bundle on X_T. This makes Bun_r into a presheaf of groupoids: this means that pullbacks and compositions of maps are compatible in some reasonable way.

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2.1. **Stacks.** It is a classical result that if you have an étale covering $\{U_i \to S\}$ (open covering in the analytic topology) and vector bundles $\mathcal{E}_i$ on X_{U_i} that "agree" on overlaps, then there exists a vector bundle $\mathcal{E}$ on X_S such that $\mathcal{E}|_{X_{U_i}} = \mathcal{E}_i$. We say that $\{\mathcal{E}_i\}$ descends to $\mathcal{E}$. In other words, vector bundles glue. So Bun_r is a "sheaf of groupoids", or a *stack* [FGI+05], with respect to the étale topology.

2.2. **Algebraic stacks.** There are certain conditions that one wants a stack to satisfy so that it is sufficiently geometric, i.e., behaves like a scheme. For stacks whose associated groupoids have large automorphism groups, the notion is called an *algebraic*, or Artin, stack [LMB00, Sta].

First, note that any sheaf of sets is a stack. So by the Yoneda embedding, any scheme can be considered as stack. We will use S to denote both the scheme and the corresponding sheaf. Let $\mathcal{Y}$ be a stack $\mathbf{Sch}_{et}^{\mathrm{op}} \to \mathbf{Gpd}$. By a 2-categorical version of Yoneda, a map $S \to \mathcal{Y}$ is equivalent to an object of $\mathcal{Y}(S)$. For $S_i \in \mathbf{Sch}$ and $y_i \in \mathcal{Y}(S_i)$ for $i = 1, 2$, we define the stack fibred product $S_1 \times_{\mathcal{Y}} S_2 : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Gpd}$ by

$$(S_1 \underset{\mathcal{Y}}{\times} S_2)(T) = \{f_i : T \to S_i,\, \alpha : f_1^*(y_1) \simeq f_2^*(y_2)\}$$

We say that a stack is *schematic* if it is isomorphic to a scheme (via Yoneda). Now $\mathcal{Y}$ is algebraic if:

(1) For any $S_i \in \mathbf{Sch}$, the fibred product $S_1 \times_{\mathcal{Y}} S_2$ is schematic[1]
(2) There exists a scheme U mapping to $\mathcal{Y}$ such that for any $S \in \mathbf{Sch}$, the base change $U \times_{\mathcal{Y}} S \to S$ is a smooth surjective map of schemes.

So now we can state the main result:

Theorem 1. *The stack* Bun_r *is algebraic.*

3. $B\mathrm{GL}_r$

Before considering vector bundles on a family of curves, let's consider a simpler problem: We define presheaf $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Gpd}$ by $B\mathrm{GL}_r(S)$ to be the groupoid of vector bundles of rank r on S (compare with Bun_r: which is vector bundles over X_S).

Theorem 2. *The presheaf* $B\mathrm{GL}_r$ *is an algebraic stack. The map from a point* $\cdot \to B\mathrm{GL}_r$ *corresponding to* $\mathbf{C}^r$ *is smooth and surjective: in fact for* $\mathcal{E} \in B\mathrm{GL}_r(S)$, *we have* $\cdot \times_{B\mathrm{GL}_r} S \simeq \mathrm{Isom}_S(\mathcal{O}^r, \mathcal{E})$ *over* S.

As a toy example, let us compute $\cdot \times_{B\mathrm{GL}_r} \cdot$. This stack sends S to an automorphism of $\mathcal{O}_T^r$, i.e., an element of $\mathrm{GL}_r(T)$. Therefore the fibred product is just GL_r in this case. For general $\mathcal{E} \in B\mathrm{GL}_r(S)$, we trivialize $\mathcal{E}$ locally, and then we must check that the GL_r glue together properly to give $\mathrm{Isom}_S(\mathcal{O}^r, \mathcal{E})$.

Since $\cdot \to B\mathrm{GL}_r$ is surjective, we can think of it as a quotient $[\cdot/\mathrm{GL}_r]$ where $\cdot$ has trivial G-action. More generally, we can consider stacks $[Z/G]$ where Z is a scheme with some G-action and G is an algebraic group. One can then describe relationships between these quotients when you change the space and/or the group.

[1] We actually only require the fibred product to be representable by an algebraic space, but our fibred products will satisfy this nicer condition.

4. Hom as a scheme

For a stack $\mathcal{Y}$, we can define another stack $\mathrm{Hom}(X, \mathcal{Y})$ sending S to the groupoid $\mathcal{Y}(X_S)$. Notice that with this definition, we get $\mathrm{Bun}_r = \mathrm{Hom}(X, B\mathrm{GL}_r)$. So before thinking about $\mathrm{Hom}(X, \mathcal{Y})$ in general, let's consider the easier problem of when $\mathcal{Y} = Y$ is a scheme: then $\mathrm{Hom}(X, Y) = \mathrm{Hom}(X_S, Y) = \mathrm{Hom}_X(X_S, X \times Y)$. This motivates the following definition: given a map of schemes $Z \to X$, let $\mathrm{Sect}(X, Z)$ be the sheaf of sets sending $S \mapsto \mathrm{Hom}_X(X_S, Z)$.

Theorem 3. *If $Z \to X$ is quasi-projective, then $\mathrm{Sect}(X, Z)$ is representable by a disjoint union of quasi-projective schemes over $\mathbf{C}$.*

This theorem is where much of the work goes into proving condition (1) of algebraicity for Bun_r. The proof involves using Grothendieck's Hilbert schemes [FGI+05]. When $Z \to X$ is projective, the idea is that a section $X_S \hookrightarrow Z_S$ corresponds uniquely to a closed subscheme, so $\mathrm{Sect}(X, Z)$ is some subscheme of the Hilbert scheme.

5. Presentation of Bun_r

Once we have condition (1) of algebraicity, all that is left is (2) to find a presentation of Bun_r, i.e., some scheme U, which will actually be locally of finite type over $\mathbf{C}$, with a smooth surjective morphism to Bun_r. This involves thinking about trivializations of vector bundles and Hilbert polynomials. As with any kind of representability result, this will again involve the Quot scheme.

Fix a very ample line bundle $\mathcal{O}(1)$ on X. We define the substack

$$\mathcal{U}_n \subset \mathrm{Bun}_r$$

with $\mathcal{U}_n(S)$ consisting of the vector bundles $\mathcal{E}$ on X_S such that $H^1(X_s, \mathcal{E}_s(n)) = 0$ and $\mathcal{E}_s(n)$ is generated by global sections, where $\mathcal{E}_s = \mathcal{E} \otimes k(s)$ is the pullback to the fiber $X_s := X_{\mathrm{Spec}\, k(s)}$. (These conditions are essentially to ensure base change and cohomology are compatible.) By Serre's theorem on quasi-coherent sheaves on projective space, these $\mathcal{U}_n$ form an open covering of Bun_r, so it suffices to look at each $\mathcal{U}_n$. To save notation, we'll just consider $\mathcal{U} := \mathcal{U}_0$. Note that $H^0(X_s, \mathcal{E}_s)$ is a finite dimensional $k(s)$-vector space. By the upper semi-continuity theorem, the dimension is locally constant in s. So $\mathcal{U} = \sqcup \mathcal{U}^d$ where d is this dimension of the space of global sections. Fix d and consider the stack

$$\mathcal{Y} \to \mathcal{U}^d$$

where $\mathcal{Y}(S)$ has objects $(\mathcal{E}, x_1, \ldots, x_d)$ consisting of a vector bundle $\mathcal{E}$ and sections $x_i \in H^0(X_S, \mathcal{E})$ such that the sections restrict to a basis on each fiber. Since $\mathcal{U}^d$ essentially just forgets about the choice of a basis, the fibers of $\mathcal{Y} \to \mathcal{U}^d$ are just GL_d. (In fact, we have $\mathcal{U}^d = [\mathcal{Y}/\mathrm{GL}_d]$.) Now $\mathcal{Y}$ has just enough structure on it that it is representable by a quasi-projective $\mathbf{C}$-scheme! So if we do this over all n, d, we get a smooth surjection $\sqcup \mathcal{Y}_n^d \twoheadrightarrow \mathrm{Bun}_r$. This proves condition (2) and hence Theorem 1.

Example 4. If $r = 1$, then Bun_1 is the Picard stack. Since X always has a k-point, we in fact have an isomorphism $B\mathbf{G}_m \times \mathrm{Pic}\, X \simeq \mathrm{Bun}_1$. This case is special since $\mathrm{GL}_1 = \mathbf{G}_m$ is abelian. So Bun_r can be seen as a non-abelian generalization of the Picard stack.

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